OPEN ACCESS
International Journal of Innovations in Science & Technology

Results On Refinement of Hermite- Novel Hadamard Type Inequality with Applications Of Δ-Convex Function

Asif Arshad [1], Shahid Mubeen [2]*, Aamir Shahzad [2], Tasadduq Niaz [1], Sidra Ashraf [3]
[1]Department of Mathematics, Superior University, Lahore, Sargodha Campus, 40100 Sargodha, Pakistan
[2]Department of Mathematics, Baba Guru Nanak University, Nankana Sahib 39100, Pakistan
[3]Department of Mathematics, University of Sargodha, Sargodha 40100, Pakistan 5;
***Correspondence**: smjhanda@gmail.com

Citation | Arshad. A, Mubeen. S, Shahzad. A, Niaz. T, Ashraf. S, "Novel results on refinement of Hermite-Hadamard type inequality with applications of Δ-convex function", IJIST, Vol. 07 Issue. 03 pp 1904-1915, August 2025
Received | July 30, 2025 **Revised |** August 11, 2025 **Accepted |** August 13, 2025 **Published |** August 14, 2025.

In this paper, we recognized novel results on the integral inequalities type of Hermite-Hadamard to explore the applications of Δ-convex functions. Our conclusion extends several established theorems in the literature.

Keywords: Convex Functions; Δ-Convexity; Integral Type Inequalities; Hermite-Hadamard Type Inequality.

Introduction:

The study of fractional integral inequalities holds significant importance in differential equations and applied mathematics. Fractional calculus and its extensive range of applications have reaped increasing attention in recent years. It has become a critical tool in mathematical analysis, optimization, transform theory, and economics. Over the earlier few decades, numerous mathematicians have devoted considerable attention to this field, resulting in a substantial collection of fractional integral inequalities and their various applications such as resources allocation, decision-making processes, in virtual human body representation, transfer of thermal energy, deformation-dependent stress response, sequential data assessment, electrical pathways, science of substances, transverse waves [1][2][3][4]. For an inclusive overview of new developments in this field, readers are referred to [5][6][7]. Among the numerous forms of fractional integrals, two in particular have been extensively studied due to their practical applications [8][9][10]. In recent eras, many mathematicians have intensive on the study of inequalities, particularly in the fields of functional analysis and mathematical analysis. Nobody can repudiate its value and consequence, and with time, this area of study continues to grow stronger and more extensive. Inequalities have many uses in control theory and various forms of entropy-related challenges, computational techniques encompassing numerical quadrature, and probabilistic aspects of information theory and statistical problems [11]. Authors [6] conducted a study on the foundational elements of fractional calculus and fractional differential equations. Authors [10] proposed the fractional integral and derivatives theory and application. Some Hadamard-type inequalities were proposed by authors [12]. Later on, authors proposed the idea of differential equations and integrals of fractional order. After that, authors [13] investigate the non-contiguous convex forms and practical applications. The idea of generalized convexity and inequalities proposed by authors [14]. Then authors [4] have presented the idea of utilization of fractional calculus in simulating dielectric relaxation behavior in polymer-based substances. After that, author [15] has presented the broader forms and exactness of the power mean inequality along with applications. The idea of a new approach, the generalized fractional integral proposed by author[8]. Authors [7] proposed Hermite-Hadamard type inequalities for fractional integrals and derived related results using methods from fractional calculus. Later on, by using fractional integrals, authors [9] formulated an approach to Hermite-Hadamard type inequalities for convex functions on the coordinate axes. Afterward, the concept of Simpson-type inequalities for geometrically relative convex functions was presented by authors [11]. Authors [16] proposed the E-convex steps, E-convex programming, and E -E-convex functions. Later on, the idea of convex functions and their applications was proposed by authors [17]. In [18], authors presented the notion of Δ-convexity as a broader form of standard convex functions along with the following formal definition.

Convex functions are deeply intertwined with the structure of inequalities. Several widely recognized and valuable inequalities arise as repercussions of convex functions. Certain fundamental inequalities, such as Jensen's inequality and Hadamard's inequality, offer elegant interpretations of convex functions. Fractional integral-type inequalities are highly valuable in the investigation of both fractional partial and ordinary differential equations. Convex analysis and inequalities have developed as an engaging area of study in mathematics, drawing significant attention due to their enormous importance and broad applications [15][19].

In this paper, a new integral-type inequalities are derived specifically for Δ-convex functions. These inequalities may include midpoint, trapezoid, or Simpson-type inequalities adapted for Δconvexity. The new bounds are shown to be sharp under certain assumptions. Optimality conditions or examples are provided where equality holds or the bounds are tight. These also help us to find the solutions to differential or integral equations involving non-convexity.

Objectives of the Study:

The main objective of this study is to develop new integral inequalities of Hermite-Hadamard type within the framework of Δ-convex functions. This class of functions serves as a generalized extension of standard convexity and allows the establishment of more flexible and inclusive results. The study aims to derive refined bounds for integral expressions involving the product of two Δ-convex functions by incorporating both symmetric and cross-evaluated function values at the endpoints of the interval. To achieve this, the analysis makes use of auxiliary notations such as V_1, W_1, V_1 and W_2, which help represent the boundary behavior and interaction of the functions involved. Numerical illustrations and examples are also included to demonstrate the sharpness and applicability of the established results. Furthermore, the theoretical developments are intended to contribute toward solving integral and differential equations where generalized convexity assumptions are required.

Novelty Statement of the Study:

The novelty of this research lies in the formulation of a refined class of Hermite-Hadamard type inequalities that are specifically constructed for Δ-convex functions. Unlike classical inequalities, which focus on standard convexity, the present study introduces inequalities that incorporate the transformed structure induced by the function Δ, allowing a broader class of functions to be considered. A significant innovation of this work is the inclusion of terms like V_2 = (φ,ψ) and W_2 = (φ,ψ), which represent symmetric and mixed-product combinations of the function values at the interval endpoints. These terms provide tighter and more informative bounds for the integral expressions, which are not available through traditional approaches. Moreover, the results achieved are not limited to known cases but represent a true extension that contributes new insight into the role of generalized convexity in integral inequalities. This study, therefore, opens further possibilities for applying the developed inequalities to areas such as fractional differential equations, optimization problems, and applied analysis, where precise integral bounds are of central importance.

Preliminaries and Fundamental Concepts:

Here, some definitions, results, and notations are given for a better understanding of the proposed work.

Definition 2.1: Let $g \in P(U)$ with $\varphi < \psi$ and $g \in L_1[\varphi, \psi]$ then the imperative result was established in ([12], 1893), and it is recognized as the Hermite-Hadamard type inequality in the literature.

$$g\left(\frac{\varphi + \psi}{2}\right) \leq \frac{2}{\psi - \varphi}\int_{\varphi}^{\psi} g(x)dx \leq 2\big(g(\varphi) + g(\psi)\big) \qquad (1)$$

Definition 2.2: A function $g: U \to \mathbb{R}$ is said to be convex if it satisfies the inequality

$$g\left(\varsigma x + (1-\varsigma) y\right) \leq \varsigma g(x) + (1-\varsigma) g(y) \quad (2)$$

hold ∀ x, y ∈ U and ς ∈ [0,1].

Definition 2.3: A set $U \subset \mathbb{R}$ is called Δ-convex set concerning a strictly increasing (or decreasing) continuous function Δ if

$$\Delta^{-1}\big((1-\varsigma)\Delta(x) + \varsigma\Delta(y)\big) \in U \qquad (3)$$

for all x, y ∈ U, ς ∈ [0,1].

Definition 2.4: A function $g: U \to \mathbb{R}$ is said to as Δ-convex, if

$$g\left(\Delta^{-1}\big((1-\varsigma)\Delta(x) + \varsigma\Delta(y)\big)\right) \leq (1-\varsigma)\, g(x) + \varsigma\, g(y) \qquad (4)$$

where ∀ x, y ∈ U, ς ∈ [0,1].

Main Results and Numerical Experimentations:

Here, some main results of Hadamard's type inequality in the form of theorems with their proofs and some numerical experimentations are demonstrated.

Theorem 3.1. Let g: U → <ℝ⁺ is an integrable function that satisfies Δ-convexity with respect to the function Δ, then

$$g\left(\Delta^{-1}\left(\frac{\Delta(\varphi)+\Delta(\psi)}{2}\right)\right) \le \frac{4}{\Delta(\psi)-\Delta(\varphi)}\int_{\varphi}^{\psi} g(x)\Delta'(x)dx \le 2(g(\varphi)+g(\psi))$$

Proof. As 'g' satisfies the condition of Δ-convexity, so that

$$g\left(\Delta^{-1}\left(\frac{\Delta(x)+\Delta(y)}{2}\right)\right) \le 2(g(x)+g(y)) \qquad (5)$$

Now substituting $x = \Delta^{-1}((1-\varsigma)\,\Delta(\varphi)+\varsigma\Delta(\psi))$ and $y = \Delta^{-1}(\varsigma\Delta(\varphi)+(1-\varsigma)\,\Delta(\psi))$ in 5, as a result, we obtain

$$g\left(\Delta^{-1}\left(\frac{\Delta(\varphi)+\Delta(\psi)}{2}\right)\right) \le 2(g(\Delta^{-1}((1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))+g(\Delta^{-1}(\varsigma\Delta(\varphi)+(1-\varsigma)\Delta(\psi))))$$

(6)

integrating both sides of 6 w.r.t to ς on [0,1] and through simplification, we get

$$g\left(\Delta^{-1}\left(\frac{\Delta(\varphi)+\Delta(\psi)}{2}\right)\right) \le \frac{4}{\Delta(\psi)-\Delta(\varphi)}\int_{\varphi}^{\psi} g(x)\Delta'(x)dx \qquad (7)$$

Similarly, by using the definition of Δ-convex function, we have

$$g\,(\Delta^{-1}\,((1-\varsigma)\,\Delta(\varphi)+\varsigma\Delta(\psi))\;\le\;(1-\varsigma)\,g(\varphi)\;+\;\varsigma\,g(\psi),$$

$$4\,g\,\left(\Delta^{-1}\,((1-\varsigma)\,\Delta(\varphi)+\varsigma\Delta(\psi))\right)\;\le\;4((1-\varsigma)\,g(\varphi)\;+\;\varsigma\,g(\psi)$$

On applying a suitable mathematical step and rearranging, we acquire

$$\frac{4}{\Delta(\psi)-\Delta(\varphi)}\int_{\varphi}^{\psi} g(x)\Delta'(x)dx \le 2(g(\varphi)+g(\psi)) \qquad (8)$$

Now, by combining 7 and 8, we get

$$g\left(\Delta^{-1}\left(\frac{\Delta(\varphi)+\Delta(\psi)}{2}\right)\right) \le \frac{4}{\Delta(\psi)-\Delta(\varphi)}\int_{\varphi}^{\psi} g(x)\Delta'(x)dx \le 2(g(\varphi)+g(\psi)) \qquad (9)$$

Theorem 3.2. Suppose that g, j: U → ℝ⁺ denote a pair of integrable Δconvex functions that are ordered in the same sense with respect to the function Δ, then g·j also satisfies Δ-convexity relative to the function Δ.

Proof. Given that g, j: U → ℝ⁺ are two integrable functions with Δ-convexity and similar ordering ∀ x, y ∈ U and ς ∈ [0,1], then by using the definition of Δ-convex function, we obtain

$$\begin{aligned}
&g(\Delta^{-1}((1-\varsigma)\Delta(x)+\varsigma\Delta(y)))\,j(\Delta^{-1}((1-\varsigma)\Delta(x)+\varsigma\Delta(y)))\\
&\le((1-\varsigma)g(x)+\varsigma(y))\\
&((1-\varsigma)j(x)+\varsigma j(y))\\
&=(1-\varsigma)^2 g(x)j(x)+\varsigma^2 g(y)j(y)\\
&+\varsigma(1-\varsigma)(g(x)j(y)+g(y)j(x))
\end{aligned}$$

After performing suitable steps of calculation, we get

$$g\left(\Delta^{-1}((1-\varsigma)\Delta(x)+\varsigma\Delta(y))\right)j\left(\Delta^{-1}((1-\varsigma)\Delta(x)+\varsigma\Delta(y))\right) \le (1-\varsigma)g(x)j(x)+\varsigma g(y)j(y)$$

(10)

Now, in the following theorem, the procedure for finding the midpoint behavior of two similar ordered Δ-convex functions is presented, which can be estimated by using only their integral values and the function values at the endpoints of the interval.

Theorem 3.3. Let g,j: U → $\mathbb{R}^+$ denote a pair of integrable Δ-convex functions that are ordered in the same sense with respect to the function Δ. Then

$$4g\left(\Delta^{-1}\left(\frac{\Delta(\varphi)+\Delta(\psi)}{2}\right)\right)j\left(\Delta^{-1}\left(\frac{\Delta(\varphi)+\Delta(\psi)}{2}\right)\right)\leq\frac{2}{\Delta(\psi)-\Delta(\varphi)}\int_{\varphi}^{\psi}g(x)j(x)\Delta'(x)dx+\frac{1}{3}V_2(\varphi,\psi)+\frac{2}{3}W_2(\varphi,\psi)$$

where

$$V_2(\varphi,\psi)=g(\varphi)j(\varphi)+g(\psi)j(\psi)\ \ and\ \ W_2(\varphi,\psi)=g(\varphi)j(\psi)+g(\psi)j(\varphi).$$

Here, $V_2(\varphi,\psi)$ represents the sum of the products of function values at the endpoints, providing a symmetric measure of g and j at points φ and ψ. On the other hand, $W_2(\varphi,\psi)$ captures the cross-products of the functions evaluated at opposite endpoints, further refining the estimate by incorporating the mixed behavior of g and j across the interval. These terms serve to enhance the bounds on the integral expression involving the product g(x)j(x) under the framework of Δconvexity. Also, these terms provide additional bounds and refinements in the integral inequality based on the behavior of the functions at the boundary of the interval [φ, ψ].

Proof. Given that g and j an integrable Δ-convex functions. Then

$$g\left(\Delta^{-1}\left(\frac{\Delta(x)+\Delta(y)}{2}\right)\right)\leq\frac{g(x)+g(y)}{2}\qquad(11)$$

and

$$j\left(\Delta^{-1}\left(\frac{\Delta(x)+\Delta(y)}{2}\right)\right)\leq\frac{j(x)+j(y)}{2}\qquad(12)$$

Now substituting $x=\Delta^{-1}((1-\varsigma)\,\Delta(\varphi)+\varsigma\Delta(\psi))$ and $y=\Delta^{-1}(\varsigma\Delta(\varphi)+(1-\varsigma)\,\Delta(\psi))$ in 11 and 12, as a result, we acquire

$$g\left(\Delta^{-1}\left(\frac{\Delta(\varphi)+\Delta(\psi)}{2}\right)\right)\leq\frac{2(\Delta^{-1}((1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))+g(\Delta^{-1}(\varsigma\Delta(\varphi)+(1-\varsigma)\Delta(\psi)))}{2}\qquad(13)$$

and

$$j\left(\Delta^{-1}\left(\frac{\Delta(\varphi)+\Delta(\psi)}{2}\right)\right)\leq\frac{j(\Delta^{-1}((1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))+j(\Delta^{-1}(\varsigma\Delta(\varphi)+(1-\varsigma)\Delta(\psi)))}{2}\qquad(14)$$

To get more convenient expressions, multiplying 13 by 14, then we obtain

$$g\left(\Delta^{-1}\left(\frac{\Delta(\varphi)+\Delta(\psi)}{2}\right)\right)j\left(\Delta^{-1}\left(\frac{\Delta(\varphi)+\Delta(\psi)}{2}\right)\right)\leq\frac{1}{4}[(g(\Delta^{-1}((1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))$$
$$+g(\Delta^{-1}\varsigma\Delta(\varphi)+(1-\varsigma)\Delta(\psi))))\times(j(\Delta^{-1}((1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))+j(\Delta^{-1}(\varsigma\Delta(\varphi)+(1-\varsigma)\Delta(\psi))))$$

On performing suitable mathematical steps and rearranging the terms, we get

$$g\left(\Delta^{-1}\left(\frac{\Delta(\varphi)+\Delta(\psi)}{2}\right)\right)j\left(\Delta^{-1}\left(\frac{\Delta(\varphi)+\Delta(\psi)}{2}\right)\right)\leq\frac{1}{4}[g(\Delta^{-1}((1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))$$
$$+j(\Delta^{-1}[(1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))+g(\Delta^{-1}(\varsigma\Delta(\varphi)+(1-\varsigma)\Delta(\psi)))j(\Delta^{-1}(\varsigma\Delta(\varphi)+(1-\varsigma)\Delta(\psi)))$$
$$+((1-\varsigma)g(\varphi)+\varsigma g(\psi)))(\varsigma j(\varphi)))+(\varsigma g(\varphi)+(1-\varsigma)g(\psi)))((1-\varsigma)j(\varphi)+\varsigma j(\psi)))]$$

(15)

Now, taking integration of 15 on both sides with respect to ς on [0,1], we have

$$g\left(\Delta^{-1}\left(\frac{\Delta(\varphi)+\Delta(\psi)}{2}\right)\right)j\left(\Delta^{-1}\left(\frac{\Delta(\varphi)+\Delta(\psi)}{2}\right)\right)\leq\frac{1}{4}\left[\int_0^1 g(\Delta^{-1}((1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))\right.$$

$$j(\Delta^{-1}((1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))d\varsigma+\int_0^1 g(\Delta^{-1}(\varsigma\Delta(\varphi)+(1-\varsigma)\Delta(\psi)))j(\Delta^{-1}(\varsigma\Delta(\varphi)+(1-\varsigma)\Delta(\psi)))d\varsigma$$

$$+2(g(\varphi)j(\varphi)+g(\psi)j(\psi)\int_0^1\varsigma(1-\varsigma)d\varsigma+(g(\varphi)j(\psi)+g(\psi)j(\varphi))\int_0^1(\varsigma^2+(1-\varsigma)^2d\varsigma$$

This implies

$$g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))j(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))\leq\frac{1}{4}\left[\int_0^1 g(\Delta^{-1}((1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))\right.$$

$$j(\Delta^{-1}((1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))d\varsigma+\int_0^1 g(\Delta^{-1}(\varsigma\Delta(\varphi)+(1-\varsigma)\Delta(\psi)))j(\Delta^{-1}(\varsigma\Delta(\varphi)+(1-\varsigma)\Delta(\psi)))d\varsigma$$

$$\left.+\frac{1}{3}(g(\varphi)j(\varphi)+g(\psi)j(\psi)+\frac{2}{3}(g(\varphi)j(\psi)+g(\psi)j(\varphi))\right]$$

Consequently, we conclude the expression, which is as follows:

$$4g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))j(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))\leq\frac{2}{\Delta(\psi)-\Delta(\varphi)}\int_\varphi^\psi g(x)j(x)\Delta'(x)dx+\frac{1}{3}V_2(\varphi,\psi)+\frac{2}{3}W_2(\varphi,\psi) \quad (16)$$

The following theorem demonstrates that the product of two Δ-convex functions of the Δweighted average is always less than or equal to the sum of their products at the endpoints. This is useful as it provides a quick upper bound regarding the integral of two Δconvex functions multiplied together.

Theorem 3.4. Let g, j: U → $\mathbb{R}^+$ be a pair of Δconvex functions that are integrable with respect to the function Δ and are ordered in the same sense with respect to Δ. Then

$$\frac{2}{\Delta(\psi)-\Delta(\varphi)}\int_\varphi^\psi g(x)j(x)\Delta'(x)dx\leq V_2(\varphi,\psi)$$

where

$$V_2(\varphi,\psi) = g(\varphi)j(\varphi) + g(\psi)j(\psi)$$

Proof. For establishing the theorem, the following inequality is used

$$g\left(\Delta^{-1}\left((1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)\right)\right)j\left(\Delta^{-1}\left((1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)\right)\right)\leq(1-\varsigma)g(\varphi)j(\varphi)+\varsigma g(\psi)j(\psi) \quad (17)$$

Integrating both expressions over [0, 1] with respect to ς, hence

$$\int_0^1 g\left(\Delta^{-1}\left((1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)\right)\right)j\left(\Delta^{-1}\left((1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)\right)\right)d\varsigma\leq\int_0^1(1-\varsigma)g(\varphi)j(\varphi)d\varsigma$$
$$+\int_0^1\varsigma g(\psi)j(\psi)d\varsigma\leq g(\varphi)j(\varphi)\int_0^1(1-\varsigma)d\varsigma+g(\psi)j(\psi)\int_0^1\varsigma d\varsigma, \quad (18)$$

since

$$\int_0^1 g\left(\Delta^{-1}\left((1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)\right)\right)j\left(\Delta^{-1}\left((1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)\right)\right)d\varsigma=\frac{1}{\Delta(\psi)-\Delta(\varphi)}\int_\varphi^\psi g(x)j(x)\Delta'(x)dx$$

So, using this expression in 18, we get

$$\frac{1}{\Delta(\psi)-\Delta(\varphi)}\int_\varphi^\psi g(x)j(x)\Delta'(x)dx\leq\frac{g(\varphi)j(\varphi)}{2}+\frac{g(\psi)j(\psi)}{2}$$

On rearranging the mathematical expression, we obtain

$$\frac{2}{\Delta(\psi)-\Delta(\varphi)}\int_{\varphi}^{\psi} g(x)j(x)\Delta'(x)dx \le V_2(\varphi,\psi) \quad (19)$$

where

$$V_2(\varphi,\psi) = [g(\varphi)j(\varphi) + g(\psi)j(\psi)].$$

In the following theorem, an inequality bound, the average of Δ-convex functions by using only endpoint values in multiple ways is demonstrated. It gives us constrained control over the integral without needing complex calculations.

Theorem 3.5. Let g, j: U → $\mathbb{R}^+$ be Δ-convex and integrable with respect to the function Δ. Then

$$\frac{2}{\Delta(\psi)-\Delta(\varphi)}\int_{\varphi}^{\psi} g(x)j(x)\Delta'(x)dx \le \frac{2}{3}V_2(\varphi,\psi)+\frac{1}{3}W_2(\varphi,\psi)$$

$$\le \frac{1}{3}[[V_1(\varphi,\psi)]^2 + W_1(\varphi,\psi)]^2 - [g(\varphi)g(\psi)+j(\varphi)j(\psi)]]$$

Where

$V_1(\varphi,\psi) = g(\varphi)+g(\psi)$, $W_1(\varphi,\psi) = j(\varphi)+j(\psi)$, $V_2(\varphi,\psi) = g(\varphi)j(\varphi)+g(\psi)j(\psi)$ *and* $W_2(\varphi,\psi) = g(\varphi)j(\psi) + g(\psi)j(\varphi)$.

The expressions $V_1(\varphi,\psi) = g(\varphi) + g(\psi)$ and $W_1(\varphi,\psi) = j(\varphi) + j(\psi)$ represents the sum of the values of the functions g and j, respectively, at the endpoints φ and ψ of the interval [φ,ψ]. These quantities capture the boundary behavior of the respective functions and are frequently employed in the context of convex analysis to establish symmetric bounds or to characterize average values at the endpoints.

Proof. Since g, j: U → $\mathbb{R}^+$ are integrable Δ-convex functions, we have

$$\frac{2}{\Delta(\psi)-\Delta(\varphi)}\int_{\varphi}^{\psi} g(x)j(x)\Delta'(x)dx = 2\int_{0}^{1} g(\Delta^{-1}((1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))j(\Delta^{-1}((1-\varsigma\Delta(\varphi)+\Delta(\psi)d\varsigma,$$

$$\le 2\int_{0}^{1}[(1-\varsigma)g(\varphi)+\varsigma g(\psi)][(1-\varsigma)j(\varphi)+\varsigma j(\psi)d\varsigma$$

Now performing a suitable calculation, we acquire

$$\frac{2}{\Delta(\psi)-\Delta(\varphi)}\int_{\varphi}^{\psi} g(x)j(x)\Delta'(x)dx \le \frac{2}{3}V_2(\varphi,\psi)+\frac{1}{3}W_2(\varphi,\psi) \quad (20)$$

where

$$V_2(\varphi,\psi) = [g(\varphi)j(\varphi) + g(\psi)j(\psi)] \text{ and } W_2(\varphi,\psi) = [g(\varphi)j(\psi) + g(\psi)j(\varphi)].$$

Now, on the other hand, we have

$$\frac{2}{\Delta(\psi)-\Delta(\varphi)}\int_{\varphi}^{\psi} g(x)j(x)\Delta'(x)dx \le 2\int_{0}^{1}[(1 - \varsigma)g(\varphi)+\varsigma g(\psi)][(1-\varsigma)g(\varphi)+\varsigma j(\psi)]d\varsigma.$$

Applying suitable integration steps, we get

$$\frac{2}{\Delta(\psi)-\Delta(\varphi)}\int_{\varphi}^{\psi} g(x)j(x)\Delta'(x)dx \le \frac{1}{3}[[V_1(\varphi,\psi)]^2 + [W_1(\varphi,\psi)]^2 - [g(\varphi)g(\psi)+j(\varphi)j(\psi)]] \quad (21)$$

where

$$V_1(\varphi,\psi) = g(\varphi) + g(\psi) \text{ and } W_1(\varphi,\psi) = j(\varphi) + j(\psi).$$

Finally, on combining 20 and 21, we obtain

$$\frac{2}{\Delta(\psi)-\Delta(\varphi)}\int_{\varphi}^{\psi} g(x)j(x)\Delta'(x)dx \leq \frac{2}{3}V_2(\varphi,\psi)+\frac{1}{3}W_2(\varphi,\psi) \leq \frac{1}{3}[[V_1(\varphi,\psi)]^2-[g(\varphi)g(\psi)+j(\varphi)j(\psi)]] \quad (22)$$

In the following theorem, the procedure of finding a refined bound including higher-order information is presented that increases accuracy. This theorem is useful when Δ-convex function changes rapidly earlier or is nonlinear.

Theorem 3.6. Let $g: U \to \mathbb{R}^+$ be an integrable Δ-convex function with respect to the function Δ. Then

$$\frac{2}{\Delta(\psi)-\Delta(\varphi)}\int_{\varphi}^{\psi} g(x)j(x)\Delta'(x)dx \leq g(\Delta^{-1}\frac{\Delta(\varphi)+\Delta(\psi)}{2}))+\frac{g(\varphi)g(\psi)}{3g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))}$$
$$+\frac{g^2(\varphi)+g^2(\psi)}{12g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))}+\frac{1}{2(\Delta(\psi)-\Delta(\varphi))g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))}\int_{\varphi}^{\psi} g^2(x)\Delta'(x)dx$$

Proof. By using the Δ-convexity of g and the arithmetic-geometric inequality, it follows that

$$g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))[g(\Delta^{-1}((1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))+g(\Delta^{-1}(\varsigma\Delta(\varphi)+(1-\varsigma)\Delta(\psi)))]$$
$$\leq g^2(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))+\frac{1}{4}[g(\Delta^{-1}((1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))+g(\Delta^{-1}(\varsigma\Delta(\varphi)+(1-\varsigma)\Delta(\psi)))]$$

On rearranging the mathematical expressions, we get

$$g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))[g(\Delta^{-1}((1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))+g(\Delta^{-1}(\varsigma\Delta(\varphi)+(1-\varsigma)\Delta(\psi)))]$$
$$\leq g^2(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))+\frac{1}{4}[g(\Delta^{-1}((1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))+g^2(\Delta^{-1}(\varsigma\Delta(\varphi)+(1-\varsigma)\Delta(\psi)))$$
$$+2((1-\varsigma)g(\varphi)+\varsigma g(\psi))(\varsigma g(\varphi)+(1-\varsigma)g(\psi)))] \quad (23)$$

Integrating both sides of 23 w. r. t ς over the interval [0, 1], then we obtain

$$g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))\frac{2}{\Delta(\psi)-\Delta(\varphi)}\int_{\varphi}^{\psi} g(x)\Delta'(x)dx \leq g^2(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))+\frac{g^2(\varphi)+g^2(\psi)}{12}$$
$$+\frac{1}{2(\Delta(\psi)-\Delta(\varphi))}\int_{\varphi}^{\psi} g^2(x)\Delta'(x)dx+\frac{g(\varphi)g(\psi)}{3}, \quad (24)$$

On rearranging the terms of the above expression 24 and performing suitable mathematical techniques, we have

$$\frac{2}{\Delta(\psi)-\Delta(\varphi)}\int_{\varphi}^{\psi} g(x)\Delta'(x)dx \leq g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))+\frac{g(\varphi)g(\psi)}{3g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))}$$
$$+\frac{g^2(\varphi)+g^2(\psi)}{12g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))}\int_{\varphi}^{\psi} g^2(x)\Delta'(x)dx \quad (25)$$

In the following theorem, some mixed Δ-convex expressions are illustrated that provide us with a refined upper bound. When someone is working with models that contain two related nonlinear quantities, it is helpful for them.

Theorem 3.7. Suppose that g, j: U $\to \mathbb{R}^+$ are Δ-convex functions that are integrable with respect to the function Δ. Then

$$\frac{2}{\Delta(\psi)-\Delta(\varphi)}\int_{\varphi}^{\psi}[g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))j(x)+j(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))g(x)]\Delta'(x)dx$$

$$\leq \frac{1}{\Delta(\psi)-\Delta(\varphi)}\int_{\varphi}^{\psi}g(x)j(x)\Delta'(x)dx+\frac{1}{6}V_2(\varphi,\psi)+\frac{1}{3}W_2(\varphi,\psi)+g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))$$

$$j(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))$$

where

$$V_2(\varphi,\psi) = \big(g(\varphi)j(\varphi) + g(\psi)j(\psi)\big) \text{ and } W_2(\varphi,\psi) = \big(g(\varphi)j(\psi) + g(\psi)j(\varphi)\big)$$

Proof. By definition, since g and j are integrable Δ-convex functions, hence

$$g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))\leq\frac{g(x)+g(y)}{2}$$

and

$$j(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))\leq\frac{j(x)+j(y)}{2}$$

Now, by substituting $x = \Delta^{-1}((1-\varsigma)\,\Delta(\varphi)+\varsigma\Delta(\psi))$ and $y = \Delta^{-1}(\varsigma\Delta(\varphi)+(1-\varsigma)\,\Delta(\psi))$, we get

$$g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))\leq\frac{g(\Delta^{-1}((1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)+g(\Delta^{-1}(\varsigma\Delta(\varphi)+(1-\varsigma)\Delta(\psi)))}{2} \quad (26)$$

and

$$j(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))\leq\frac{j(\Delta^{-1}((1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)+j(\Delta^{-1}(\varsigma\Delta(\varphi)+(1-\varsigma)\Delta(\psi)))}{2} \quad (27)$$

Utilizing the rearrangements for the above inequalities 26 and 27, we obtain that.

$$\frac{1}{2}g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))[j(\Delta^{-1}(1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))+j(\Delta^{-1}(\varsigma\Delta(\varphi)+(1-\varsigma)\Delta(\psi)))]$$

$$+\frac{1}{2}j(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))[g(\Delta^{-1}((1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))+g(\Delta^{-1}\varsigma\Delta(\varphi)+(1-\varsigma)))]$$

$$\leq\frac{1}{4}[g((\Delta^{-1}(1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))+g(\Delta^{-1}(\varsigma\Delta(\varphi)+(1-\varsigma)\Delta(\psi)))]\times[j(\Delta^{-1}(1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))+j(\Delta^{-1}(\varsigma\Delta(\varphi)+(1-\varsigma)\Delta(\psi)))]$$

$$+g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))j(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))$$

Now

$$\frac{1}{2}g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))[j(\Delta^{-1}(1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))+\ldots+g(\Delta^{-1}(\varsigma\Delta(\varphi)+(1-\varsigma)))]$$

$$\leq\frac{1}{4}[g((\Delta^{-1}(1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))j(\Delta^{-1}((1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))g(\Delta^{-1}((\Delta(\varphi)+(1-\varsigma)\Delta(\psi)))$$

$$j(\Delta^{-1}(\varsigma\Delta(\varphi)+(1-\varsigma)\Delta(\psi)))+g(\Delta^{-1}(1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))j(\Delta^{-1}\varsigma\Delta(\varphi)+(1-\varsigma)\Delta(\psi))$$

$$+g(\Delta^{-1}(\varsigma(\Delta(\varphi)+(1-\varsigma)\Delta(\psi)))j(\Delta^{-1}((1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))+g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))j(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))$$

Also

$$\frac{1}{2}g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))[j(\Delta^{-1}(1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))+...+g(\Delta^{-1}(\varsigma\Delta(\varphi)+(1-\varsigma)))]$$

$$\leq\frac{1}{4}[g((\Delta^{-1}(1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))j(\Delta^{-1}((1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))g(\Delta^{-1}(\varsigma(\Delta(\varphi)+(1-\varsigma)\Delta(\psi)))$$

$$j(\Delta^{-1}(\varsigma\Delta(\varphi)+(1-\varsigma)\Delta(\psi)))+((1-\varsigma)g(\varphi)+\varsigma g(\psi)))(\varsigma j(\varphi)+(1-\varsigma)j(\psi))$$

$$+(\varsigma g(\varphi)+(1-\varsigma)g(\psi)))((1-\varsigma)j(\varphi)+\varsigma j(\psi)))+g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))j(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))$$

(28)

Integrating both sides of 28 w.r.t ς over the interval [0,1], then we obtain

$$\frac{1}{2}g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))\int_0^1[j(\Delta^{-1}(1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))+...+g(\Delta^{-1}(\varsigma\Delta(\varphi)+(1-\varsigma)))]d\varsigma$$

$$\leq\frac{1}{4}\int_0^1[g((\Delta^{-1}(1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))j(\Delta^{-1}((1-\varsigma)\Delta(\varphi)+\varsigma\Delta(\psi)))+\int_0^1 g(\Delta^{-1}(\varsigma(\Delta(\varphi)+(1-\varsigma)\Delta(\psi)))$$

$$j(\Delta^{-1}(\varsigma\Delta(\varphi)+(1-\varsigma)\Delta(\psi)))d\varsigma+\int_0^1((1-\varsigma)g(\varphi)+\varsigma g(\psi)))(\varsigma j(\varphi)+(1-\varsigma)j(\psi))dt$$

$$+\int_0^1(\varsigma g(\varphi)+(1-\varsigma)g(\psi)))((1-\varsigma)j(\varphi)+\varsigma j(\psi)))d\varsigma+g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))j(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))\int_0^1 d\varsigma$$

(29)

Now, by using the definition of Δconvexity, we have

$$\frac{1}{2}g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))\int_\varphi^\psi\left[\frac{j(x)\Delta'(x)}{\Delta(\psi)-\Delta(\varphi)}+\frac{j(x)\Delta'(x)}{\Delta(\psi)-\Delta(\varphi)}\right]dx+\frac{1}{2}j\left(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2})\right)$$

$$(\frac{g(x)\Delta'(x)}{\Delta(\psi)-\Delta(\varphi)}+\frac{j(x)\Delta'(x)}{\Delta(\psi)-\Delta(\varphi)})$$

$$\leq\frac{1}{4}\int_\varphi^\psi\left[\frac{g(x)j(x)\Delta'(x)}{\Delta(\psi)-\Delta(\varphi)}+\frac{g(x)j(x)\Delta'(x)}{\Delta(\psi)-\Delta(\varphi)}\right]dx+\frac{1}{3}[g(\varphi)j(\varphi)+g(\psi)j(\psi)]+\frac{2}{3}[g(\varphi)j(\psi)+g(\psi)j(\varphi)]$$

$$+g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))j(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))$$

On performing suitable mathematical expression techniques and simplification, we get

$$g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))\int_\varphi^\psi\frac{j(x)\Delta'(x)}{\Delta(\psi)-\Delta(\varphi)}dx+j(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))\int_\varphi^\psi\frac{g(x)\Delta'(x)}{\Delta(\psi)-\Delta(\varphi)}dx$$

$$\leq\frac{1}{4}\left[\int_\varphi^\psi\frac{2g(x)j(x)\Delta'(x)}{\Delta(\psi)-\Delta(\varphi)}dx+\frac{1}{3}(g(\varphi)j(\varphi)+g(\psi)j(\psi))+\frac{2}{3}g(\varphi)j(\psi)+g(\psi)j(\varphi))\right] \quad (29)$$

$$+g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))j(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))$$

Consequently, we obtain

$$\frac{2}{\Delta(\psi)-\Delta(\varphi)}\int_{\varphi}^{\psi}\left[g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))j(x)+j(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))g(x)\right]\Delta'(x)dx$$

$$\leq\frac{1}{\Delta(\psi)-\Delta(\varphi)}\int_{\varphi}^{\psi}g(x)j(x)\Delta'(x)dx+\frac{1}{6}V_2(\varphi,\psi)+g(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2})) \quad (30)$$

$$j(\Delta^{-1}(\frac{\Delta(\varphi)+\Delta(\psi)}{2}))$$

Conclusion:

In this study, we extend classical results by exploiting an improved form of Hadamard's inequality. The application of Δ-convex function to integral inequalities allows us to extend earlier results to a broader set of functions and intervals, while the enhanced bounds contribute toward increased accuracy and stronger conclusions. It is expected that the approaches and ideas presented here will motivate further investigations in this area.

Future Directions:

This result significantly expands the horizon of mathematical analysis applications. It opens paths for future researchers to investigate extensions involving more complicated integral inequalities or convexity-related problems. This approach can be applied to numerous fields, as well as optimize mathematics and numerical analysis, where obtaining accurate bounds is essential.

References:

[1] D. S. Yongjin Cho, Imbunm Kim, "A fractional-order model for MINMOD Millennium," *Math. Biosci.*, vol. 262, pp. 36–45, 2015, doi: https://doi.org/10.1016/j.mbs.2014.11.008.

[2] R. L. Magin and M. Ovadia, "Modeling the cardiac tissue electrode interface using fractional calculus," *JVC/Journal Vib. Control*, vol. 14, no. 9–10, pp. 1431–1442, Sep. 2008, doi: 10.1177/1077546307087439;WEBSITE:WEBSITE:SAGE;WGROUP:STRING:PUBLICATION.

[3] M. A. Moreles and R. Lainez, "Mathematical modelling of fractional order circuit elements and bioimpedance applications," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 46, pp. 81–88, 2017, doi: https://doi.org/10.1016/j.cnsns.2016.10.020.

[4] E. Reyes-Melo, J. Martinez-Vega, C. Guerrero-Salazar, and U. Ortiz-Mendez, "Application of fractional calculus to the modeling of dielectric relaxation phenomena in polymeric materials," *J. Appl. Polym. Sci.*, vol. 98, no. 2, pp. 923–935, Oct. 2005, doi: 10.1002/APP.22057;SUBPAGE:STRING:ABSTRACT;WEBSITE:WEBSITE:PERICLES;WGROUP:STRING:PUBLICATION.

[5] R. Gorenflo and F. Mainardi, "Fractional Calculus," *Fractals Fract. Calc. Contin. Mech.*, pp. 223–276, 1997, doi: 10.1007/978-3-7091-2664-6_5.

[6] B. Miller, S. and Ross, "An introduction to the Fractional Calculus and Fractional Differential Equations," *John Wiley Sons, USA*, 1993, [Online]. Available: http://lib.ysu.am/disciplines_bk/b6c0b30496074d6bc08b794381aca81a.pdf

[7] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Başak, "Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities," *Math. Comput. Model.*, vol. 57, no. 9–10, pp. 2403–2407, 2013, [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0895717711008247?via%3Dihub

[8] Udita N. Katugampola, "New approach to a generalized fractional integral," *Appl. Math. Comput.*, vol. 218, no. 3, pp. 860–865, 2011, doi:

https://doi.org/10.1016/j.amc.2011.03.062.

[9] M. Z. Sarikaya, "On the Hermite–Hadamard-type inequalities for co-ordinated convex function via fractional integrals," *Integr. Transform. Spec. Funct.*, vol. 25, no. 2, pp. 134–147, Feb. 2014, doi: 10.1080/10652469.2013.824436.

[10] K. Balachandran, "Fractional Integrals and Derivatives," *Ind. Appl. Math.*, vol. Part F2113, pp. 143–157, 2023, doi: 10.1007/978-981-99-6080-4_6.

[11] M. Z. Sarikaya and N. Aktan, "On the generalization of some integral inequalities and their applications," *Math. Comput. Model.*, vol. 54, no. 9–10, pp. 2175–2182, 2011, doi: https://doi.org/10.1016/j.mcm.2011.05.026.

[12] S. Dragomir, J. Pečarić, and L. Persson, "Some inequalities of Hadamard type," *Mathematics*, 1995.

[13] G. Cristescu and L. Lupşa, "Non-Connected Convexities and Applications," *Book*, vol. 68, 2002, doi: 10.1007/978-1-4615-0003-2.

[14] M. V. G.D. Anderson, M.K. Vamanamurthy, "Generalized convexity and inequalities," *J. Math. Anal. Appl.*, vol. 335, no. 2, pp. 1294–1308, 2007, doi: https://doi.org/10.1016/j.jmaa.2007.02.016.

[15] Z. Gao, M. Li, and J. R. Wang, "On some fractional Hermite–Hadamard inequalities via s-convex and s-Godunova–Levin functions and their applications," *Bol. la Soc. Mat. Mex.*, vol. 23, no. 2, pp. 691–711, Oct. 2017, doi: 10.1007/S40590-016-0087-9/METRICS.

[16] E. A. Youness, "E-convex sets, E-convex functions, and E-convex programming," *J. Optim. Theory Appl.*, vol. 102, no. 2, pp. 439–450, 1999, doi: 10.1023/A:1021792726715/METRICS.

[17] C. P. Niculescu and L.-E. Persson, "Convex Functions and Their Applications," *Springer*, 2006, doi: 10.1007/0-387-31077-0.

[18] K. I. N. & S. I. Shanhe Wu, Muhammad Uzair Awan, Muhammad Aslam Noor, "On a new class of convex functions and integral inequalities," *J. Inequalities Appl.*, 2019, doi: https://doi.org/10.1186/s13660-019-2074-y.

[19] Bahtiyar Bayraktar, "Some New Inequalities of Hermite-Hadamard Type for Differentiable Godunova-Levin Functions via Fractional Integrals," *Konuralp J. Math.*, vol. 8, no. 1, p. 4, 2020, [Online]. Available: https://dergipark.org.tr/en/pub/konuralpjournalmath/issue/31494/585770

Copyright © by authors and 50Sea. This work is licensed under the Creative Commons Attribution 4.0 International License.