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Computational Insights into Lissajous Curves: Wave Superposition, Pattern Formation, and Visualization

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Lissajous curves provide a fundamental graphical representation of wave superposition and phase relationships in oscillatory systems. In this study, we present a computational investigation of two-dimensional and three-dimensional Lissajous curves, along with a brief contextual overview of the evolution of computational approaches. The curves are generated through direct evaluation of analytical expressions to visualize a wide range of patterns produced by two mutually perpendicular simple harmonic waves. The formation and geometric characteristics of these curves are systematically examined as functions of frequency ratios, phase differences, and amplitudes. High-resolution visualizations demonstrate how variations in these parameters govern pattern symmetry, closure, and spatial complexity in both two and three dimensions. The three-dimensional extension, in particular, provides deeper insight into the structural richness and dynamic behavior of coupled oscillatory systems. This work contributes to the computational exploration of Lissajous curve formation by presenting an accessible, reproducible, and efficient visualization framework that is independent of any specific programming environment. The approach enhances the interpretation of wave interference and superposition phenomena and serves as a valuable tool for physics education, signal analysis, and pattern formation studies. The results are exact within the limits of the underlying analytical formulation.

Keywords: Computational Analysis, Lissajous Curves, Superposition, Signal Visualization, Oscillatory Systems, Simple Harmonic Waves

Introduction:

The study of waves and oscillations is a fascinating concept in physics and engineering. A wave is a disturbance that propagates through matter or space, transferring energy from one point to another. Waves are analyzed using the wave equation, which describes how oscillations travel over time [1]. The interaction of waves produces diverse phenomena like interference, diffraction, stationary waves, beats, Lissajous curves, etc. [1][2][3]. When two or more mutually perpendicular sinusoidal waves are superimposed, they form Lissajous curves, also known as Bowditch curves [1][2][3][4]. This family of curves was first studied by Nathaniel Bowditch in 1815, but a detailed investigation was done in 1857 by Jules Antoine Lissajous [2][3][4]. Atoine Lissajous employed tuning forks oriented at right angles, together with reflecting mirrors, to generate the Lissajous curves. The geometrical construction of these curves was a laborious task, so in 1869, Edward C. Pickering introduced a device that produced the curves automatically [5]. Hubert Airy employed the Blackburn compound pendulum and successfully reproduced many Lissajous curves in 1881 [6]. The mechanical device that uses swinging pendulums to produce complex geometric and artistic patterns is called a harmonograph. In a 2023 paper, Arturo Gallozzi and Rodolfa Maria Strolla discussed its various types [7]. The study of Lissajous curves has been conducted mechanically by numerous researchers over time; however, computational investigations of this phenomenon only began after the advent of computers and programming languages [4].

In this paper, we focus specifically on the computational study of Lissajous curves, concentrating on the development and evolution of computational methods used to analyze and visualize these curves. Some computational studies used computing languages, including FORTRAN, BASIC, and PASCAL [8]. B. A. Smith used BASIC to generate Lissajous curves, exemplifying the application of programming in early computational research [9]. During the late 1990s and the early 21st century, a new generation of programming languages emerged, including Perl, Java, C++, Python, and JavaScript [8]. In some other cases, MATLAB and Python are used across diverse applications [10].

Clifford A. Pickover generated spherical Lissajous curves using a graphic supercomputer in 1991 [11]. In 2003, Marita Barabash used MATLAB to visualize cycloids, billiards, and Lissajous by using irrational numbers [12]. Rosliana Eso and colleagues generated 3D Lissajous curves with Excel spreadsheets in 2017 [13]. In the same year, Zaheer Uddin et al utilized Microsoft Excel to generate Lissajous curves and graphs of the damped harmonic oscillator [14]. In 2018, Himawan Putranta and Heru Kuswanto used the spreadsheet to draw 2D Lissajous curves in their paper "Spreadsheet for physics: Lissajous curve" [15]. In the year 2021, Tiandong Li et al used Python to study three-dimensional spatial curves briefly, but a detailed study of Lissajous curves was done by Deyvid W da M Pastana and Manuel E Rodrigues in the paper "Using Mathematica software to graph Lissajous figures" [4][16]. In 2022, Vijay et al. used open-source mathematical software to plot wave equations and Lissajous curves [17]. In 2023, Adrina Radu et al described didactic tools developed from Excel spreadsheets for the visualization of Lissajous curves generated from the composition of mutually perpendicular harmonic oscillations [18]. The results of these studies can be enhanced through the use of more powerful computing tools.

Currently, there has been a remarkable advancement in the field of computing, both in hardware and software. Researchers employ image processing and machine learning techniques to simulate and visualize natural phenomena, including tsunamis, wildfires, and other complex events [19]. J. Josiach Steckenrider et al. discussed the application of Lissajous curves in aerial search operations, and research in this novel field is actively underway [20]. In 2025, Zhihan Su examined Lissajous curves based on non-harmonic and nonstationary dynamics [21]. There is a limited availability of systematic computational studies that comprehensively explore Lissajous curve formation, particularly with respect to parameter-driven pattern variation and three-dimensional representations. To enhance reproducibility, we clarify that the resultant graphs and images can be obtained using any standard computational environment capable of evaluating trigonometric functions and rendering parametric plots. The emphasis of this work is on the analytical formulation and parameter-driven pattern formation, rather than on dependence on a particular software platform or implementation-specific settings.

The paper is organized as follows: Section 2 covers the generation and the superposition of two SHWs by varying amplitudes, phase angles, and frequencies with their mathematical description. In section 3, we have presented diverse LC depending on frequency ratios and phase difference angles. All the organizational details are also presented in the form of a flowchart for clarity.

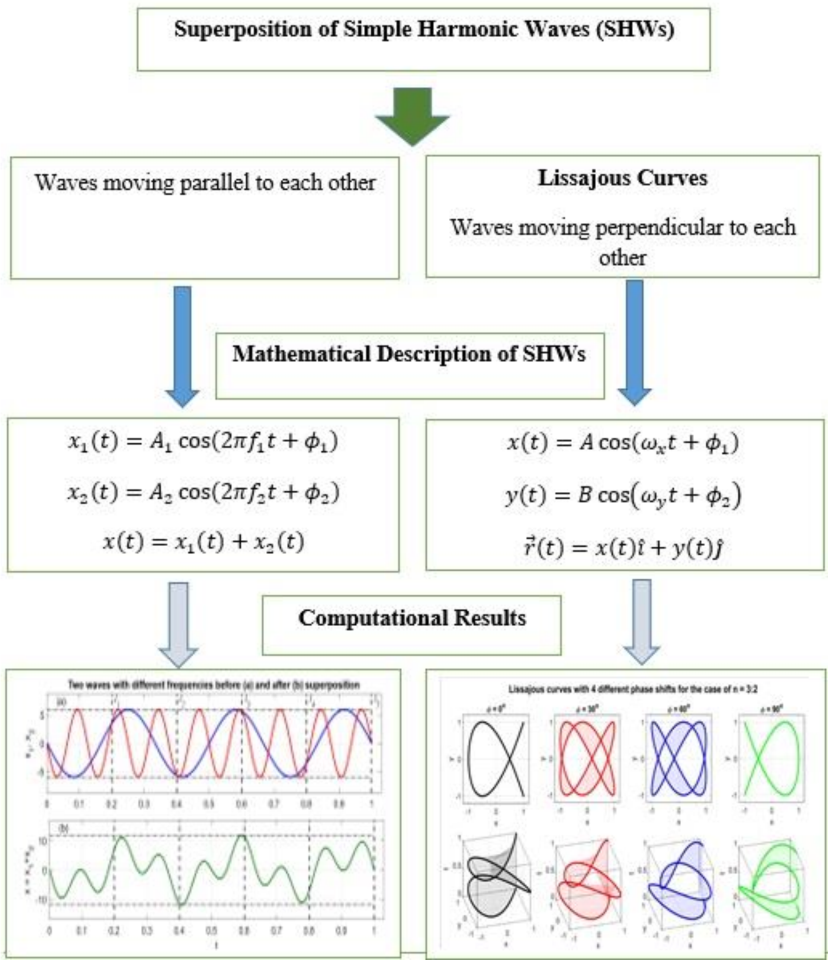


Figure 1. Flowchart illustrating the study of superposition of simple harmonic waves (SHWs).

There are two types of superposition of harmonic waves: waves moving parallel to each other and waves moving perpendicular to each other, the latter forming Lissajous curves. The mathematical description of parallel waves is simple and one-dimensional, whereas the description of Lissajous curves is two-dimensional, as illustrated in Figure 1.

Materials and Methods:

In this section, we discuss materials and methods. The methodology includes specification of the parameter ranges (e.g., frequency ratios, phase differences, and amplitudes) used to obtain smooth, high-resolution visualizations. We initially consider mathematical equations of simple sinusoidal waves. These waves are described by characteristics such as amplitude, frequency, and phase angle [1]. Two waves can differ in a few or all of the aforementioned characteristics. For illustration, in Figure 2, a set of two different waves having different: (a) amplitudes, (b) phases, and (c) frequencies is shown.

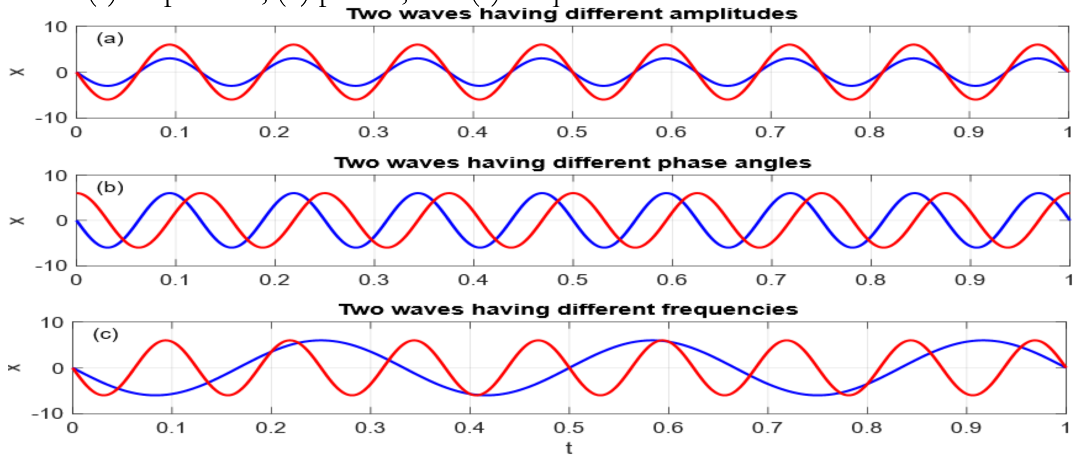


Figure 2. A set of two waves having different amplitudes (a), phases (b), and frequencies (c).

Mathematical Formulation of Superposition of Waves:

This section presents the necessary mathematical equations for the superposition of waves. A sinusoidal wave, for which the general equation of displacement is given by

$$x(t) = A\cos(2\pi ft + \phi)\,, (1)$$

In equation 1, $T = \frac{1}{f}$ is the period of a sinusoidal wave, 'x' is the displacement from the mean position, 'A' is the maximum displacement called amplitude, 'f' is the linear frequency, and 'ϕ' is the initial phase angle. Also, $\omega = 2\pi f$ is the angular frequency.

Superposition of two SHWs propagating in the same direction:

Consider two simple harmonic waves (SHWs) moving along the x direction.

$$x_1(t) = A_1\cos(2\pi f_1 t + \phi_1)\,, (2)$$

$$x_2(t) = A_2\cos(2\pi f_2 t + \phi_2)\,, (3)$$

When these SHWs are simultaneously superimposed at a point on a particle, the principle of superposition determines the resultant motion.

$$x(t) = x_1(t) + x_2(t), (4)$$

Superposition of two SHWs propagating at right angles to each other:

Consider two SHWs, one is moving along the x direction, and the other is moving along the y direction.

$$x(t) = A\cos(\omega_x t + \phi_1)\,, (5)$$

$$y(t) = B\cos\left(\omega_y t + \phi_2\right), (6)$$

When the mutually perpendicular SHWs act simultaneously on a particle, the particle follows a two-dimensional path. According to the superposition principle, the particle's position is obtained from the superposition of the two waves.

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}, (7)$$

The paths or trajectories in the xy plane of any particle are known as Lissajous curves or Bowditch curves [1][3]. The form of these trajectories (LC) depends on frequencies, initial phase angles, and amplitudes [1]. Several mechanical methods for generating Lissajous curves have been reported by [22] but the curves in this study are generated computationally.

In terms of the linear frequency f, we can rewrite equations (5) and (6) as

$$x(t) = A\cos(2\pi f_x t), (8)$$

$$y(t) = B\cos\left(2\pi f_y t + \phi\right), (9)$$

The combination of equations (8) and (9) results in

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}-\frac{2xy}{AB}\cos\phi\ =\sin^2\phi\,,(10)$$

This is the general equation of an ellipse for all possible values of A, B, and ϕ, where ϕ is the initial phase difference of the second wave, **A** is the amplitude of the x-wave, and **B** is the amplitude of the y-wave [1].

Special Cases:

If$\phi=(2k+1)\frac{\pi}{2}, where\ k\in Z$, the equation (10) can be reduced to the standard equation of an ellipse [1],

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}=1,\qquad \phi=(2k+1)\frac{\pi}{2},(11)$$

Furthermore, if $A=B,$ it becomes a circle (a special case of an ellipse) with the center at the origin.

If $\phi=k\pi,$ then another special case appears as

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}-\frac{2xy}{AB}(-1)^k=0,$$

$$y=(-1)^k\frac{B}{A}x,\quad \phi=k\pi,(12)$$

This is the equation of a straight line.

Thus, for equal frequencies ($n=1:1$) and different values of amplitude and phase differences, ϕ, we get the simplest Lissajous curves in the form of ellipses (for all cases), straight lines for quadrantal angles like $\phi=k\pi, k\in Z$, and the special case of an ellipse as a circle is formed for equal amplitudes and $\phi=(2k+1)\frac{\pi}{2}$.

Theorem 1: If the frequency ratio n is a rational number $(n=p/q), where\ p$ and q are positive integers. Then, the Lissajous curve is closed, and the trajectories are periodic [1][2][3][4]

Theorem 2: If frequency ratio n is an irrational number $(n=P/Q), where\ P$ and Q are not integers. Then, the Lissajous curve is not closed, and the trajectories are not periodic [1][2][3][4].

Results and Discussion:

In this section, we present the computer-generated graphs, Lissajous curves, using the mathematical equations mentioned in Section 2.

Superposition of two SHWs propagating along the same line:

Here, we demonstrate the computer simulation of the superposition of waves. Figures 3 through 5 present three cases: Case I shows waves with different amplitudes, Case II with different phases, and Case III with different frequencies. The waves are divided into five segments (frames) as indicated by vertical lines (l_1, l_2, l_3, l_4 andl_5). It is to illustrate that the resultant curve segment is equal to the sum of the waves enclosed in the corresponding segment or frame.

Case I: Two waves having different amplitudes

For Case I, Figure 3a shows two waves with different amplitudes, while Figure 3b shows the resultant wave after the superposition (sum of the two waves). Figure 3b illustrates that the net amplitude is the sum of individual propagating waves. The frequency and phase stay the same. The resultant equation of the superposition of two waves $x_1$and x_2 is given by

$$x_1=A_1\,cos(2\pi ft+\phi)\ \text{ and }\ x_2=A_2\,cos(2\pi ft+\phi)$$

$$x=x_1+x_2=x_m cos(2\pi ft+\phi),(13)$$

Where $x_m=A_1+A_2$ is the amplitude of the resultant wave.

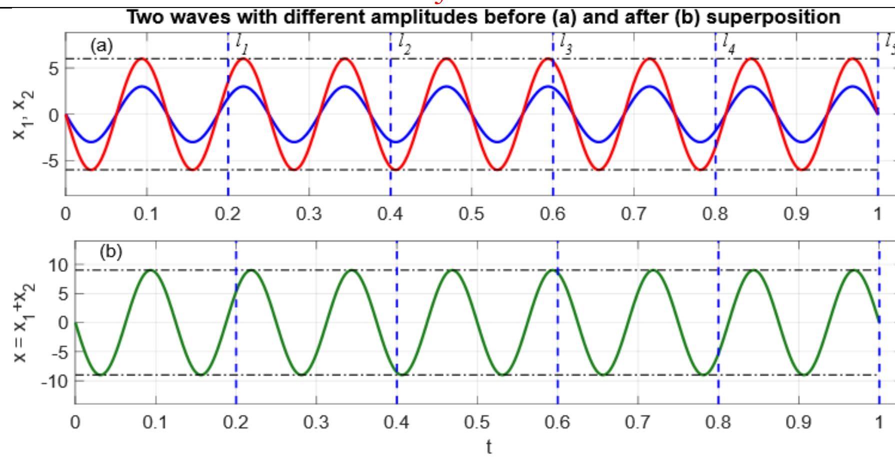


Figure 3. Two waves having (a) different amplitudes before superposition and (b) a resultant wave after superposition. Here $A_1 = 3, A_2 = 6, f_1 = f_2 = 8,\ \phi_1 = \phi_2 = \frac{\pi}{2}, 0 \leq t \leq 1$

Case II: Two waves having different phase angles:

For case II, two waves having different phases are shown in Figure 4a, and a resultant wave Figure 4b after superposition (sum of the two waves) are shown for comparison. Figure 4b shows that the superposition wave has a greater amplitude. However, unlike case I, the frequency and phase have changed in this case.

The resultant equation of the superposition of two waves $x_1$and x_2 is given by

$$x_1 = A\ cos(2\pi ft + \phi_1)$$
$$x_2 = A\ cos(2\pi ft + \phi_2)$$
$$x = x_1 + x_2 = x_m cos(2\pi ft + \phi_{av}), (14)$$

where $x_m = 2Acos\left(\frac{\phi_1 - \phi_2}{2}\right)$ is the amplitude of the resultant wave and $\phi_{av} = \frac{\phi_1 + \phi_2}{2}$ is the initial phase angle [1].

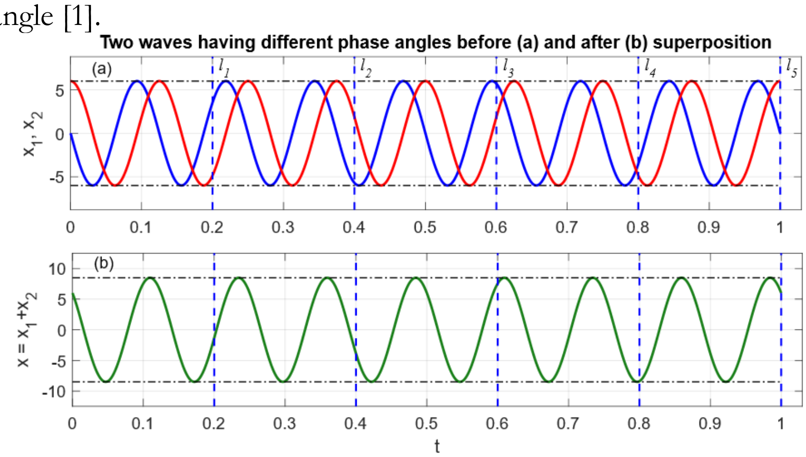


Figure 4. Two waves having (a) different phase angles before superposition and (b) a resultant wave after superposition. Here $A_1 = A_2 = 6, f_1 = f_2 = 8, \phi_1 = \frac{\pi}{2}, \phi_2 = 2\pi, 0 \leq t \leq 1$

Case III: Two waves having different frequencies

For case III, two waves having different frequencies are shown in Figure 5a and a resultant wave Figure 5b after superposition (sum of the two waves) are shown for comparison. As shown in Figure 5b, the resultant wave has a larger amplitude. However, the frequency and phase are changed in this case as well. The resultant equation of the superposition of two waves having different frequencies is given by

$$x_1 = A\ cos(2\pi f_1 t + \phi) \text{ and } x_2 = A\ cos(2\pi f_2 t + \phi)$$

$$x(t) = x_1 + x_2 = x_m(t)\cos(2\pi f_{av} t + \phi)\,, (15)$$

Where $x_m(t) = 2A\cos\left(2\pi \frac{f_1 - f_2}{2} t\right)$ is the resultant amplitude, which is time-dependent and $f_{av} = \frac{f_1 + f_2}{2}$ is the frequency of the resultant wave [1].

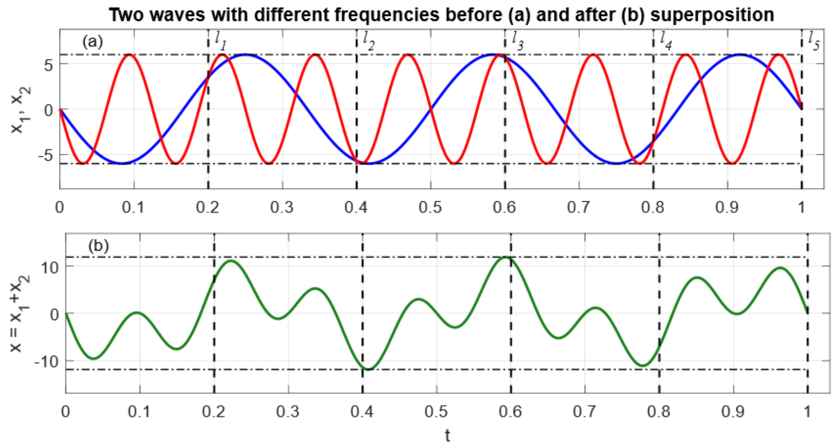


Figure 5. Two waves having (a) different frequencies before superposition and (b) a resultant wave after superposition. Here$A_1 = A_2 = 6, f_1 = 3, f_2 = 8, \phi_1 = \phi_2 = \frac{\pi}{2}, 0 \leq t \leq 1$

Superposition of two SHWs propagating at right angles to one another:

In this section, we used equations 10, 11, and 12, which describe the motion of two SHWs propagating at a right angle to one another, to generate Lissajous curves. The curves are in the form of straight lines, ellipses, circles, or some intricate shapes.

Phase diagrams for two cosine waves having equal frequencies:

Consider two cosine waves along the x direction and the y direction having the frequency ratio, $n = f_y : f_x = 1:1$, i.e., both waves have the same frequency. Here, x-wave is plotted against y-wave. This results in the Lissajous curves in the form of an ellipse. A straight line is formed if the two waves are in phase or $k\pi$ radian out of phase, where $k \in Z$. A special case of the ellipse as a circle is formed if the two waves are $(2k+1)\frac{\pi}{2}$ radian out of phase, and the amplitude ratio is $m = B:A = 1:1$, i.e., both waves have the same amplitude.

Lissajous curves, for various phase differences ϕ, were plotted in Figure 6. The phase-difference angles for y-wave start from 0^o and end at 330^o with a step size of30^0. We created a phasor diagram of Lissajous curves for equal frequencies ($n = 1:1$) and equal amplitude ratios $(m = 1:1)$ and in 12 subplots.

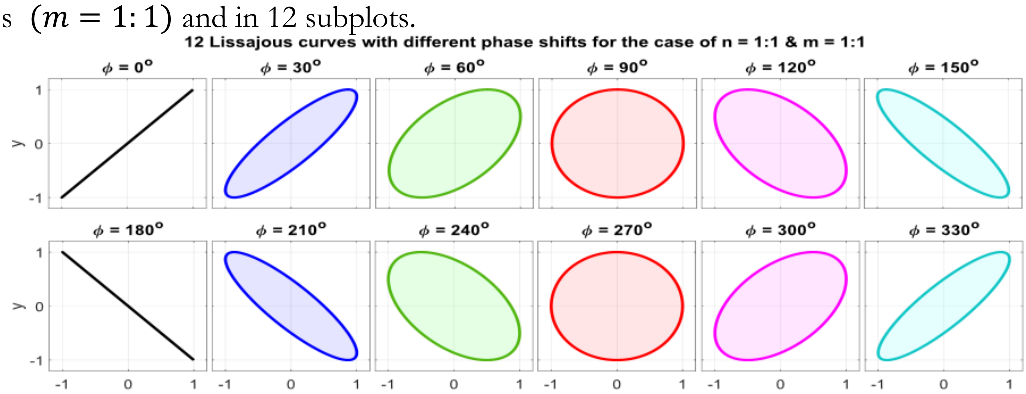
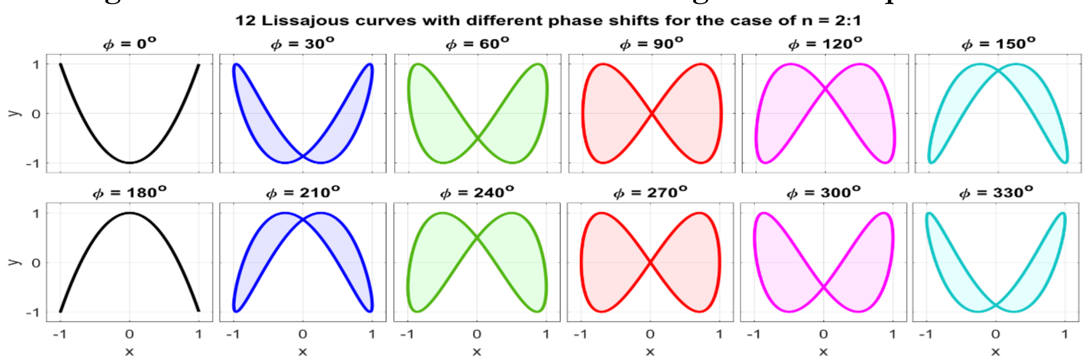


Figure 6. Lissajous curves formed by two mutually perpendicular 1D harmonic waves along the x and y directions. Phase diagram of twelve Lissajous curves for the case of frequency ratio $n = 1:1$ and amplitude ratios $m = 1:1$, with$\phi = [0^0 : 30^0 : 330^0]$.

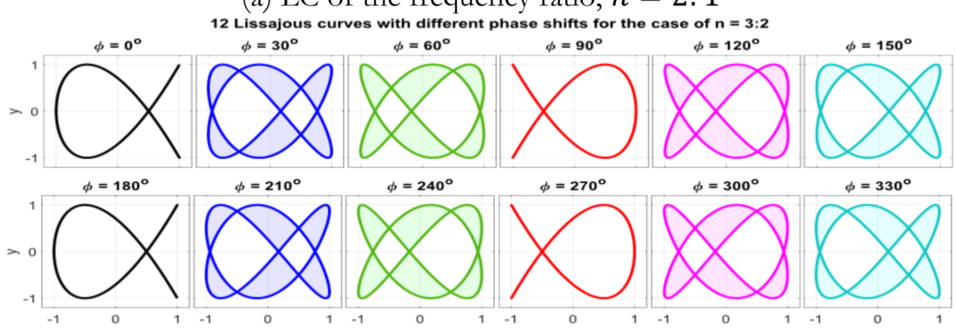
In the first row of Figure 6, as we move from left to right, the first curve is a straight line with a phase difference $0^{0,}$ i.e., SHWs are in phase, and later are ellipses with $30^0, 60^0, 120^0, \text{and } 150^0$ out of phase. In the second row from left to right, the first curve is again a straight line with 180^0 out of phase, and later are ellipses with $210^0, 240^0, 300^0, 330^0$ out of phase. At 90^0 and 270^0 phase differences, the LC are circles. Since the frequency ratio is rational and the cosine waves share the same period, the Lissajous figures are closed, smooth, and symmetrical. If the frequency ratio were irrational, the curves would be neither closed nor symmetrical [1].

In the next sections, we discussed LC with equal amplitudes ($A = B$) but different frequency ratios, $n = f_y : f_x$, and different phase angles, ϕ.

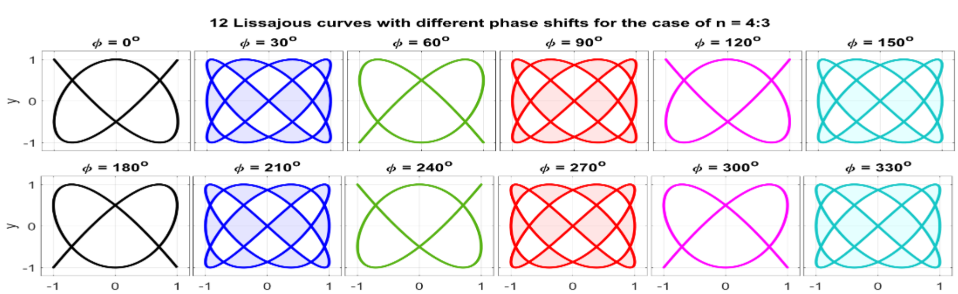
Phase diagrams for the LC of two cosine waves having different frequencies:



(a) LC of the frequency ratio, $n = 2:1$



(b) LC of the frequency ratio, $n = 3:2$

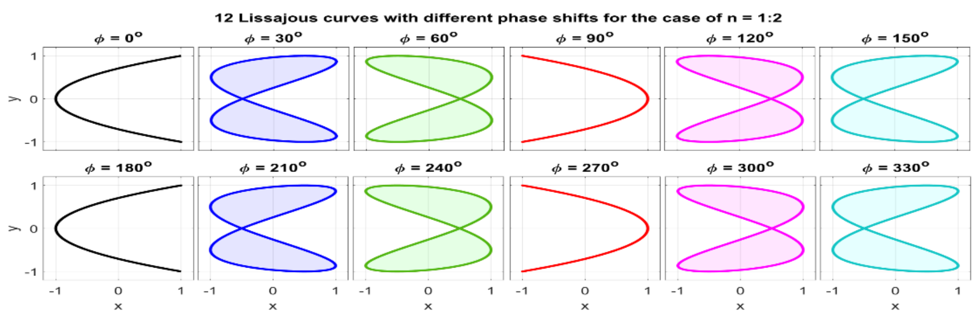


(c) LC of the frequency ratio, $n = 4:3$

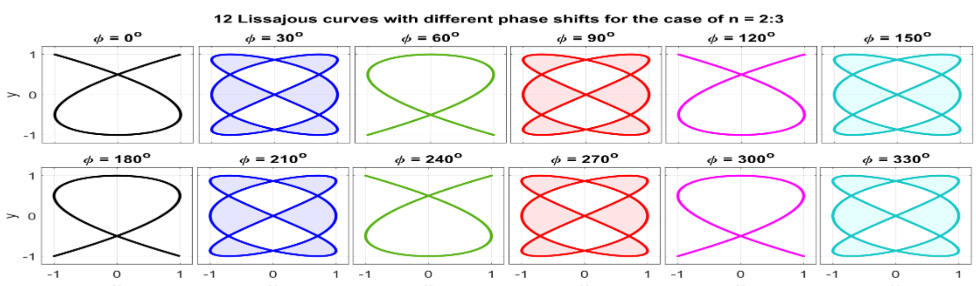
Figure 7. Lissajous curves for the cases of the same amplitudes and different frequency ratios, $n = f_y : f_x$ as (a) $n = 2:1$, (b) $n = 3:2$, (c) $n = 4:3$, and phase differences, $\phi = [0^0 : 30^0 : 330^0]$.

When the frequency ratio is a non-unity rational number, LC forms increasingly complex closed shapes, as shown in Figure 7. If the two SHWs' frequency ratios do not match exactly, the LC shapes change continuously, producing more intricate patterns. Lissajous curves strongly depend on frequency ratios ($n = f_y : f_x$, i.e., $f_y = nf_x$) and the phase difference angles (ϕ). In this section, we generated Lissajous curves for diverse frequency ratios. Each figure has 12 Lissajous curves in the subplots corresponding to phase shift $\phi = [0^0 : 30^0 : 330^0]$ and the implemented frequency ratios are $n = f_y : f_x = 2:1, 3:2, 4:3$.

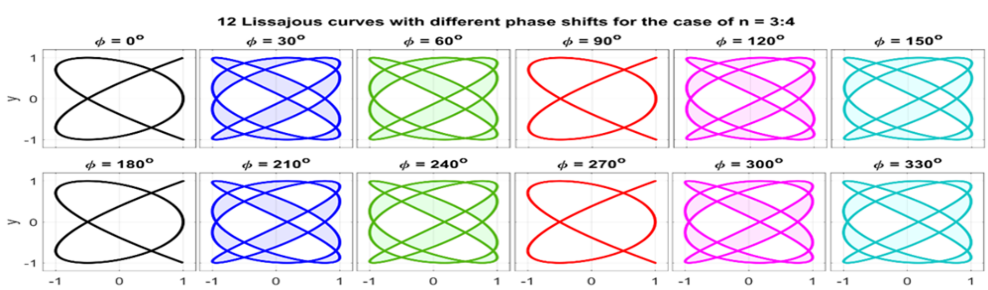
Figure. 7 shows that the superposition of two mutually perpendicular SHWs results in LC. In this part of computational analysis, the frequency of the y waves is greater than or equal to that of the x waves. If the frequencies of x and y waves are interchanged so that the frequency of y waves is less than the frequency of x waves. We observe the resultant Lissajous curves as follows.



(a) LC of the frequency ratio, n=1:2



(b) LC of the frequency ratio, n=2:3



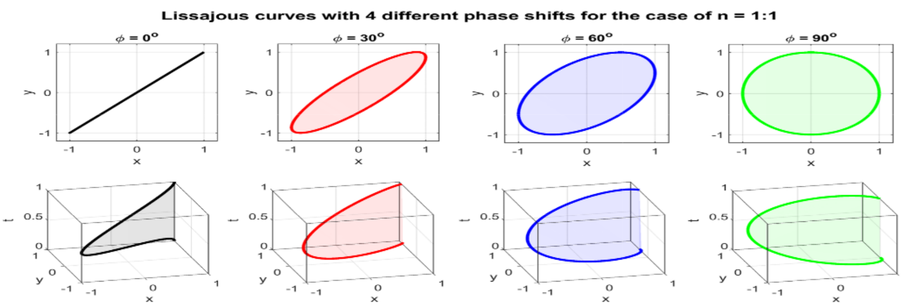
(c) LC of the frequency ratio, n=3:4

Figure 8. Lissajous curves generated by two harmonic waves along x and y directions for the case of (a) $n = 1:2$, (b) $n = 2:3$, (c) $n = 3:4$ with equal amplitudes, and the phase differences as $\phi = [0^0 : 30^0 : 330^0]$.

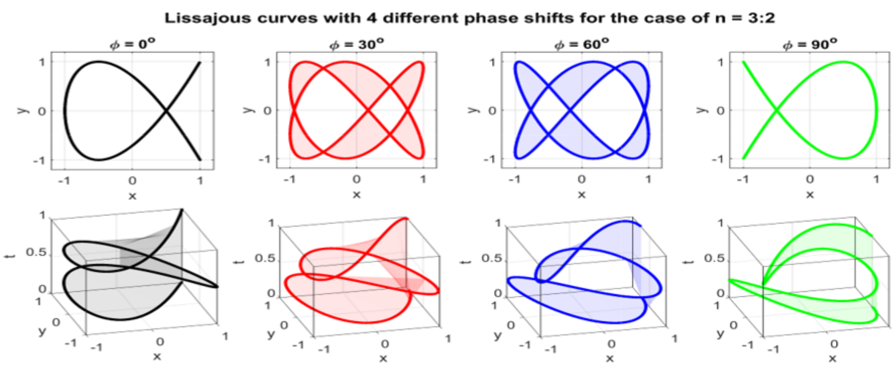
For Figures 7 and 8, non-unity rational frequency ratios were used, resulting in the curves eventually repeating and closing. The smooth, periodic nature of sine and cosine functions produces symmetrical, continuous curves, with the number of loops along each axis matching its frequency.

3D Phase diagrams for Lassijous Figures (LF) of two cosine waves having different frequency ratios:

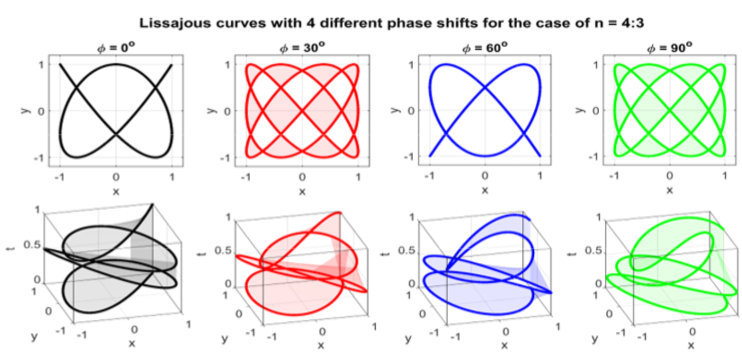
Here, we calculate the x and y waves such that y waves are determined for each phase angle: 0°, 30°, 60°, and 90°, respectively. Just for the sake of completeness, these four phase differences are used. The Lissajous curves drawn by the locus (x, y) waves with frequency ratio (a) $n = 1:1$, (b) $n = 3:2$, (c) $n = 4:3$ are shown in Figure 9$(a, b, c,)$.



(a) LF for $n = 1:1$ along with 3D trajectories



(b) LF for $n = 3:2$ along with 3D trajectories



(c) LF for $n = 4:3$ along with 3D trajectories

Figure 9. (x, y) in 2D (first rows) and (x, y, t) in 3D (second rows) Lissajous curves for the cases of (a) $n = 1:1$, (b) $n = 3:2$, (c) $n = 4:3$ with phase differences of $\phi = [0^0: 30^0: 90^0]$.

In Figure 9, the trajectories of the Lissajous curves are shown with the evolution of time by using a 3D plot in the second row of each figure. The curves are closed and periodic because the frequency ratio is still a rational number.

To enhance clarity, the next figure shows the formation of the Lissajous curve and the trajectories represented by arrows in the 2D (x, y) and 3D (x, y, t) plots.

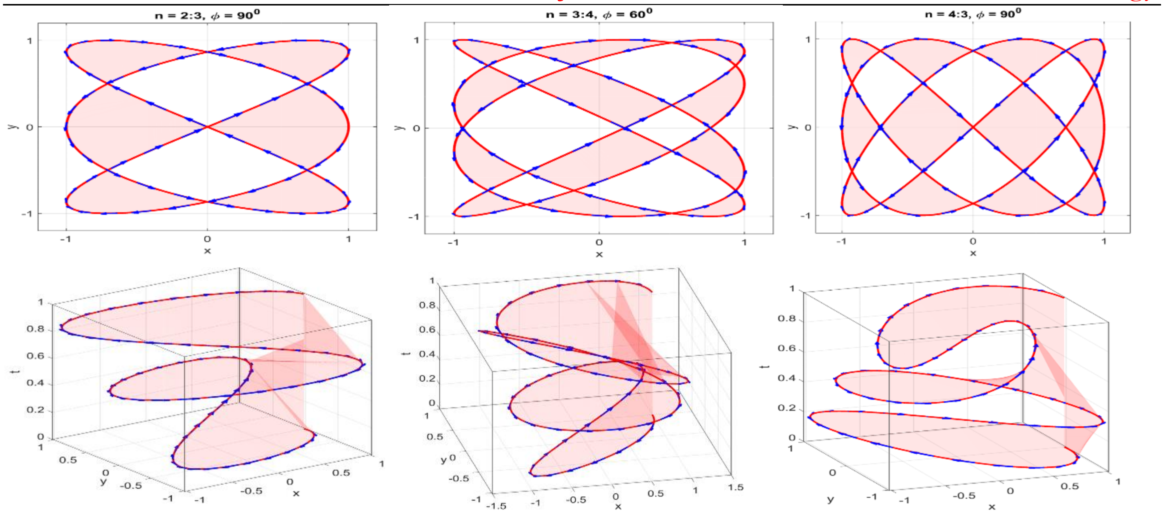


Figure 10. 2D (first row) and 3D (second row) Lissajous curves for different frequency ratios and phase shifts as $n = 2:3, \phi = 90^0$ (left column), $n = 3:4, \phi = 60^0$ (center column), $n = 4:3, \phi = 90^0$ **(right column)**. The time perimeter, t, is included in 3D plots along the z direction.

In Figure 10, Lissajous curves are shown with the evolution of time by using a 3D plot. The blue arrows show the direction of trajectories creating a smooth, symmetrical pattern with loops corresponding to the frequency values.

Lissajous curves for the case of irrational frequency ratios:

When the frequency ratio is an irrational number, then more interesting open shapes and non-periodic trajectories emerge. For instance, if $n = f_y : f_x = 2:\sqrt{3}$ (on the left side), $n = 3:\sqrt{2}$ (on the right side), for the phase difference of $\phi = 60^0$, open-shaped Lissajous curves will be generated that are not periodic, as shown in Figure 11.

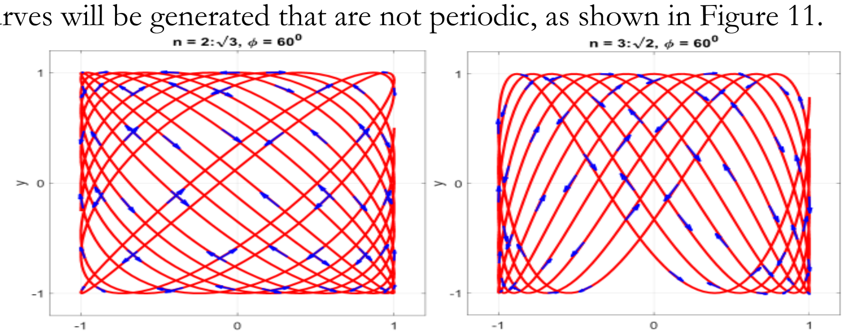


Figure 11. Lissajous curves (not closed) for the case when frequency ratios n are irrational numbers.

When the ratio of the frequencies of x and y waves is an irrational number, a complex pattern emerges, as illustrated in Figure 11, because the oscillations no longer align and there is no finite period T at which both x and y complete a full cycle simultaneously. Also, the motion never closes into a single repeating curve; instead, the trajectory continues to evolve and densely fills a region. These irrational frequency ratios are employed in modern search and surveillance systems to enable faster detection of drones and unmanned aerial vehicles (UAVs) [20]

Current and Future Research on Lissajous Curves:

Although the concept of Lissajous curves originated long ago, it continues to be widely studied due to its significant importance in engineering and technology. The analysis of Lissajous curves provides critical information about the characteristics of the signals that generate them. Current applications span physics education, electronics, optics, quantum mechanics, bio-medical signal processing, and power disturbance detection using artificial intelligence-driven image recognition, anomaly detection in rotor dynamics and vibrations, as well as more recent applications in aerial search and surveillance operations like autonomous drone search patterns. [23][24][25].

Conclusion:

This study examines the mathematical and computational details of plotting sinusoidal waves, their superposition, and the formation of Lissajous curves (LC). Phasor diagrams, depicting diverse trajectories influenced by variations in phase, frequency, and amplitude, have been generated and visualized through both two-dimensional and three-dimensional graphs. The simulations demonstrate the relationship between frequency ratios, amplitude, and the resulting shapes of LCs. When the frequency ratio is rational, the LC manifests as a closed trajectory; otherwise, it adopts an open curve. This work highlights the progress in Lissajous curve analysis driven by advances in computational tools and programming languages. The generation of LC through computational methods not only enhances learners' understanding but also offers the opportunity to visually and dynamically engage with these figures. Lissajous curves have practical applications in contemporary optics, aerial search and surveillance technologies, power quality and fault diagnosis, biomedical and signal analysis, etc. The computational results are subjected to some limitations, such as idealistic physical oscillatory models, equation accuracy, time-step selection, and three-dimensional representations are projected on a two-dimensional screen.

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