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A Meshfree Simulation of a Class of Higher-Order Fractional Differential Equations Using Radial Basis Functions and Caputo Derivative

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This paper develops an efficient radial basis function (RBF) collocation method for the numerical solution of higher-order fractional differential equations involving the Caputo derivative of order $1 < \alpha \leq 2$. The fractional derivative is evaluated using the three-point finite difference method. The proposed approach transforms the governing equation into a linear algebraic system, enabling accurate and computationally efficient numerical simulations. The accuracy and robustness of the method are validated through benchmark problems with fractional orders $\alpha = 1.25, 1.5, 1.75, 1.9, 1.95$. Numerical results demonstrate that the method achieves errors ranging from 10^{-2} to 10^{-7}, depending on the test problem, node density, and choice of RBF. Convergence analysis reveals near-exponential decay of the error as the number of nodes increases for smooth solutions, with consistent convergence observed for more challenging cases. Comparative studies show that the multiquadric RBF outperforms quintic and spline-based RBFs, yielding significantly lower errors, often by several orders of magnitude, than classical numerical methods. These findings confirm that the proposed RBF collocation framework is a reliable, accurate, and efficient tool for solving higher-order fractional differential equations arising in engineering and applied sciences.

Keywords: Radial Basis Functions, Caputo Derivative, Fractional Differential Equations, Meshfree Methods, Finite Difference Method

Introduction:

Fractional calculus represents a generalized form of classical calculus that includes non-integer orders. Over the past few decades, it many researchers have devoted considerable attention due to its wide range of applications across several disciplines of science and engineering [1][2]. Fractional calculus provides powerful analytical tools for solving integral and differential equations and offers efficient techniques for addressing physical problems involving special functions and their generalizations [3][4][5][6][7]. Fractional integral and differential equations, in which the order of differentiation or integration is non-integer, are formulated using fractional derivatives or integrals. These equations often provide more accurate representations of many physical and biological systems compared with classical integer-order differential equations. This improved modeling capability arises from their ability to incorporate non-local properties and long-term memory effects that are frequently present in complex systems.

Fractional differential equations (FDEs) serve as effective tools for describing complex phenomena that cannot be adequately modeled using traditional integer-order approaches. In general, the application of FDEs enables more comprehensive and realistic modeling of complex systems in various scientific disciplines. Due to their extensive applicability, many researchers have devoted considerable attention to the study of FDEs [8][9]. The foundational theory of fractional operators and fractional-order differential equations has been extensively developed by [1][2][3]. Their work established the theoretical framework and highlighted numerous applications of fractional operators in modeling important physical phenomena, as well as presenting various analytical and numerical techniques for solving FDEs.

FDEs have been widely applied in numerous fields like physics, engineering, finance, and economics [4]. For example, they are used to model anomalous diffusion, fractal geometry, and non-local transport phenomena. In signal processing, fractional models are used to analyze signals that exhibit long-term memory effects, such as speech and biomedical signals, enabling more accurate analysis and interpretation. In control engineering, fractional models are utilized in the design of controllers for complex systems, including robots, vehicles, and industrial processes, where they provide a more accurate representation of system dynamics and enable improved control performance. Furthermore, in financial mathematics, FDEs are employed to model the dynamics of financial markets, including stock prices and interest rates. They provide a more precise mathematical description of market behavior, facilitating improved modeling of price evolution and supporting data-driven investment strategies. The following class of FDEs is considered here [10][11]:

$$AD^{\alpha}u(t) + Bu'(t) + Cu(t) = g(t), t \in [0, T], 1 \leq \alpha \leq 2, (1)$$

with initial conditions:

$$u(0) = b_0,\ u'(0) = b_1, (2)$$

where D^{α} denotes the α-order Caputo fractional derivative, $g(t)$ is given a smooth function, and B and C are not both zero.

Due to their vast applications, FDEs were solved through numerous numerical methods. [10] presented a spline-based technique for the solution of (1). [11] proposed a fractional Adams-Bashforth-Moulton method for the solution of (1). [12] solved (1) by the difference method while translating the fractional order to a classical one. [13] applied an accurate method based on a multiscale orthonormal basis for (1). A fractional generalized Adams method is implemented by [14] to solve (1). [15] constructed a predictor-corrector method for the solution of (1). [16] solved fractional Riccati equations by a four-stage Runge-Kutta-like method.

Numerical techniques can be categorized into two main categories. The first category includes methods that require the construction of a mesh over the computational domain, such as the finite volume method and the finite difference method. The second category consists of meshless techniques, including approaches that use radial basis functions (RBFs) [17]. Although mesh-based computational methods are widely used, they often suffer from several limitations, including slow convergence rates, high computational cost, reduced accuracy, instability, spatial dependence, and difficulties in implementation for complex geometries. Moreover, mesh generation itself is a major challenge in mesh-based approaches, particularly when dealing with irregular domains, complicated geometries, or higher-dimensional problems.

In contrast, RBF-based approaches are meshless because they do not require connectivity between points within the computational domain. These methods are independent of the domain geometry, infinitely differentiable, highly accurate, and capable of achieving exponential convergence, while also eliminating the need for mesh generation.

The concept of radial basis functions was first introduced by [18] for the mathematical approximation of two-dimensional geographical surfaces. Later, in 1990, [19] proposed the RBF collocation method for solving elliptic, parabolic, and hyperbolic types of differential equations, which is now usually known as the Kansa method. In 2013, [20] extended the RBF formulations for fractional derivatives and fractional integrals. They derived Caputo and Riemann–Liouville type fractional derivatives and integrals for five classes of RBFs and applied them, together with the Kansa method, to solve different mathematical problems.

In recent years, numerous studies have employed RBF-based meshless formulations to efficiently solve different types of equations [21][22][23][24][25][26][27][28][29][30].[21] provided an efficient method for fractional stochastic integral equations using the RBF collocation method. [22] proposed an economical and flexible RBF method for the solution of fractional Laplacian visco-acoustic wave equations using collocation points in uniform and quasi-uniform layouts. [23] employed an efficient differential quadrature method along with local RBF for 2D Sobolev equations. An RBF method for spatial discretization, along with non-uniform discretization in time for the time-fractional jump model for European and American option pricing models, was presented in [24][25] developed and solved a two-term time-fractional anomalous diffusion model in mathematical physics, further demonstrating the versatility of RBF-based methods for both fractional and classical systems. [26] proposed an efficient approach for time-fractional differential equations with memory using the local RBF technique. An accurate differential quadrature-based RBF method for the fractional advection-diffusion equation is investigated in [27][28] derived an efficient computational pseudo-spectral generalized multi-quadratic RBF approach for hyper-singular integral. [29] utilized an RBF-based collocation method for a nonlinear fractional problem with memory. Recently, [30] provided an efficient method for the fractional Bagley-Torvik equation in fluid-structure systems using the RBF collocation principle. Overall, the findings confirm the robustness and capability of RBF meshless techniques for simulating numerous physical and engineering systems.

Literature Review:

Meshfree methods based on radial basis functions have gained considerable attention for solving fractional differential equations (FDEs) due to their geometric flexibility, high accuracy, and independence from mesh generation. The theoretical foundation was established by [20], who derived explicit expressions for Caputo and Riemann–Liouville fractional derivatives of several RBF types, including multiquadric, polyharmonic splines, and Gaussian kernels. Building on Kansa's pioneering collocation approach [19], numerous studies have applied RBF techniques to various fractional models. [21] developed a discrete collocation scheme for fractional stochastic integro-differential equations, while [22] combined Grünwald–Letnikov approximations with RBF collocation to solve fractional Laplacian visco-acoustic wave equations on complex domains. Employed a local RBF-based differential quadrature method for time-fractional PDEs arising in physics and engineering, demonstrating high convergence rates with modest computational cost.

Recent years have witnessed a growing interest in RBF methods for fractional models with memory and singular kernels. [25] used a local meshless method for two-term time-fractional anomalous diffusion equations, and [26] proposed an efficient RBF collocation approach for second-order fractional partial integro-differential equations. [24] introduced an RBF method with temporal graded meshes for fractional jump-diffusion option pricing, and [28] presented a fractional-order-dependent RBF meshless method for the integral fractional Laplacian. In parallel, meshfree techniques have been extended to handle variable-coefficient fractional PDEs [31], time-fractional diffusion with moving boundaries [32], and the fractional Bagley-Torvik equation [33], and to fractional stochastic differential equations [34], further demonstrating their versatility.

Despite these advances, the majority of existing RBF-based fractional solvers are designed for orders $\alpha \in (0,1)$, while systematic treatment of the higher-order range $\boldsymbol{\alpha} \in (1,2]$ remains comparatively underexplored. A unified framework that uses both infinitely smooth (e.g., multiquadric) and piecewise smooth (e.g., quintic spline) RBFs for such higher-order Caputo problems is lacking. The present work addresses these gaps by developing a meshfree RBF collocation method tailored for higher-order FDEs of the form (1)– (2), along with a comprehensive performance evaluation.

Objective:

To introduce a flexible, mesh-free radial basis function computational framework for solving a class of higher-order fractional differential equations that avoids structured meshes, enabling efficient treatment of irregular geometries and scattered data while remaining simple to implement.

To develop a meshfree RBF collocation method for solving a class of higher-order fractional differential equations with Caputo derivatives.

To achieve high numerical accuracy.

To compare the performance of different RBF types and benchmark against established numerical methods.

To analyze convergence behavior with respect to node distribution and problem parameters.

Novel Contributions:

The main contributions of this work are as follows:

Development of a unified RBF collocation framework for higher-order Caputo fractional differential equations with fractional order $\alpha \in (1,2)$.

Derivation of an efficient technique for FDEs with Caputo derivatives using the finite difference method.

Demonstration of enhanced accuracy and convergence compared to classical fractional numerical methods.

Validation of the method for higher-order fractional cases, which are rarely addressed in existing literature.

Methodology:

To develop the proposed method, we discretize the interval $[0, T]$ as $0 = t_1 < t_2 < t_3 < \cdots < t_N = T, h = t_{i+1} - t_i, i = 1,2, \dots, N-1$.

The following mathematical definitions are used to develop the proposed method.

Definition-1:

If there exists a univariate function $\phi: [0, \infty) \rightarrow \mathbb{R}$ such that $\Phi(t) = \phi(\|t\|)$, then the function $\Phi: \mathbb{R}^n \rightarrow \mathbb{R}$ is called radial, where $t \in \mathbb{R}^n$ and $\|.\|$ denotes the Euclidean distance.

Definition-2:

$\phi(s)$ is a distance-dependent function and is known as a radial basis function [27]. Furthermore, it is a one-variable continuous real-valued function. The RBFs used in the present work are as follows:

Piecewise Smooth:

$$\text{Quintic}\,\phi(s) = s^5,$$

Infinite Smooth:

$$\text{Multiquadric (MQ)}\,\phi(s) = \sqrt{s^2 + c^2},$$

c is called the shape parameter that controls both the conditioning of the system matrix and the accuracy of the solution (see [27] for details).

Definition-3:

The Caputo derivative D^α of order $\alpha, 1 \leq \alpha \leq 2,$ has the following form [35].

$$D^\alpha f(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-q)^{1-\alpha} f^{(2)}(q)dq. \quad (3)$$

Theorem-1: The Caputo fractional derivative of order $\alpha, 1 \leq \alpha \leq 2$ has the following approximation [35]:

$$D_h^\alpha f(t_k) = \frac{h^{-\alpha}}{\Gamma(3-\alpha)} \sum_{l=1}^{k-1} c_{k,l}\big(f(t_{l+2}) - 2f(t_{l+1}) + f(t_l)\big), (4)$$

where $c_{k,l} = (k-l)^{2-\alpha} - (k-l-1)^{2-\alpha}$.

Theorem-2: For $f(t) \in C^3[0,T]$, $D_h^\alpha f(t_k), 1 \leq \alpha \leq 2$ satisfies the following [35]

$$D_h^\alpha f(t_k) = D^\alpha f(t_k) + E(t_k),$$

$$\text{with } |E(t_k)| \leq \frac{2{t_k}^{2-\alpha}}{\Gamma(3-\alpha)} O(h).$$

Approximation

The RBF approximation of the function $u(t)$, is given by [27]:

$$u(t) = \sum_{j=1}^{N} \lambda_j \phi(r_j), \quad (5)$$

where $r_j = \|t - t_j\|$.

Substituting in (1), we get

$$AD^\alpha\left(\sum_{j=1}^{N} \lambda_j \phi(r_j)\right) + B\sum_{j=1}^{N} \lambda_j \phi'(r_j) + C\sum_{j=1}^{N} \lambda_j \phi(r_j) = g(t). (6)$$

Taking $t = t_k, k = 2,3, \ldots, N-1,$ in Eq. (6)

$$A\sum_{j=1}^{N} \lambda_j D^\alpha \phi(r_{kj}) + B\sum_{j=1}^{N} \lambda_j \phi'(r_{kj}) + C\sum_{j=1}^{N} \lambda_j \phi(r_{kj}) = g(t_k), \qquad (7)$$

Now using Eq. (4) in Eq. (7),

$$A\sum_{j=1}^{N}\lambda_j\frac{h^{-\alpha}}{\Gamma(3-\alpha)}\left(\sum_{l=1}^{k-1}c_{k,l}\left(\phi(r_{l+2,j})-2\phi(r_{l+1,j})+\phi(r_{l,j})\right)\right)+B\sum_{j=1}^{N}\lambda_j\phi'(r_{kj})$$
$$+C\sum_{j=1}^{N}\lambda_j\phi(r_{kj})=g(t_k).\,(8)$$

For $k=1,N$, from Eqs. (2) and (5), we have

$$\sum_{j=1}^{N}\lambda_j\phi(r_{1j})=b_0,\quad \text{and}\quad \sum_{j=1}^{N}\lambda_j\phi'(r_{1l})=b_1.\,(9)$$

The matrix form of (8) and (9) is as follows:

$$\mathbf{D}\boldsymbol{\lambda}=\mathbf{G},\,(10)$$

where $\mathbf{D}=[d_{kj}:k,j=1,2,\dots,N]$ is $N\times N$ matrix, and $\boldsymbol{\lambda}=[\lambda_j:j=1,2,\dots,N]^T$ and $\mathbf{G}=[g_k:\boldsymbol{k}=1,2,\dots,N$ such that $g_k=g(t_k),k=2,3,\dots,N-1,g_1=b_0,g_N=b_1\,]^T$ are $N\times 1$ matrices such that

$$d_{kj}=\begin{cases}\dfrac{Ah^{-\alpha}}{\Gamma(3-\alpha)}\left(\displaystyle\sum_{l=1}^{k-1}c_{k,l}\left(\phi(r_{l+2,j})-2\phi(r_{l+1,j})+\phi(r_{l,j})\right)\right)+B\phi'(r_{kj})+C\phi(r_{kj}), & \\ & k=2,3,\dots,N-1,j=1,2,\dots,N,\\ \phi(r_{1j}), & k=1,j=1,2,\dots,N,\\ \phi'(r_{1j}), & k=N,j=1,2,\dots,N.\end{cases}$$

After solving (10) for the unknown parameters λ_j by any efficient solver, the solution of (1)–(2) will be obtained from (4).

Algorithm

For the implementation of the method in Eq. (10), the following steps are given:

Selection of Nodes in the domain

Select N nodes t_l from the domain set $[0,T]$ to construct an RBF approximation

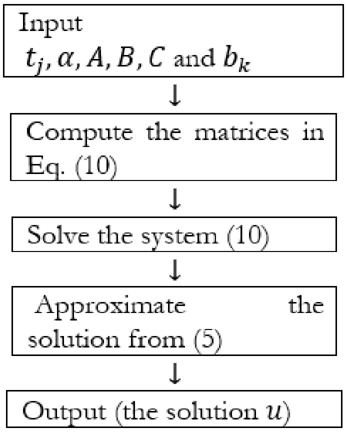
Computation of RBF Matrices

Select the RBF type and compute the elements of the matrices and vectors in Eq. (10).

Solution

Solve Eq. (10) for $\lambda_j's$ and compute the solution from Eq. (5)

The algorithm can be implemented as follows:



Numerical Experiments, Results, and Discussion:

Numerical simulations of the proposed method (10) are provided in this section. To demonstrate its efficacy, two test examples are considered on the domain [0,1] as mentioned elsewhere. The performance, accuracy, and reliability of the method are evaluated through the error norms. L_∞,L_2, pointwise absolute error, and run time (RT). Implementation of the proposed method was carried out using uniformly distributed nodes on a computer system with a Core i3 processor operating at 2.4 GHz and 2 GB RAM.

Test Example-1:

In the first example [10] we consider the FDE (1)–(2) with $A = 1, B = 0, C = 1$, $g(t) = t^2 + \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha}, u(0) = 0, u'(0) = 0$, with the exact analytical solution $u(t) = t^2$.

The proposed RBF collocation method was first applied to Example-1, which has a smooth polynomial solution $u(t) = t^2$. Tables 1-3 report the L_∞ and L_2 errors and pointwise absolute errors obtained with the multiquadric (MQ) and quintic RBFs for fractional orders $\alpha = 1.25,\ 1.5,\ 1.95$ and mesh sizes $h = 1/5, 1/10, 1/20, 1/40, 1/80, 1/160$. The results show that both RBFs yield high accuracy, with errors decreasing as the mesh is refined. For instance, with $\alpha = 1.5$ and $h = 1/40$, the MQ solution achieves $L_\infty \approx 1.35 \times 10^{-7}$ and $L_2 \approx 8.07 \times 10^{-8}$. The multiquadric RBF consistently outperforms the quintic RBF, and is also significantly more accurate than the Spline method [10].

Figure 1 displays the excellent agreement between the MQ approximate solution and the exact solution for $\alpha = 1.75$ and $h = 1/20$, with the pointwise error remaining below 5×10^{-7}. Figure 2 illustrates the approximate solutions for different α values, showing that the method captures the solution behavior accurately across the entire fractional order range.

Table 1. Error norms of MQ solutions for Example-1

		α=1.25		α=1.5		α=1.95		
h	c	L_∞	L_2	L_∞	L_2	L_∞	L_2	RT(sec)
1/5	5	6.06e-05	4.39e-05	6.64e-05	4.75e-05	7.28e-05	5.10e-05	0.0312
1/10	2.5	5.08e-07	3.52e-07	5.66e-07	3.87e-07	6.33e-07	4.19e-07	0.3033
1/20	2	3.14e-07	2.15e-07	3.45e-07	2.22e-07	4.12e-07	2.59e-07	0.0347
1/40	1.5	1.33e-07	8.74e-08	1.35e-07	8.07e-08	1.51e-07	9.79e-08	0.0408

Table 2. Error norms of Quintic solutions for Example-1

	α=1.25		α=1.5		α=1.95		
h	L_∞	L_2	L_∞	L_2	L_∞	L_2	RT(sec)
1/20	1.49e-02	9.94e-03	1.62e-02	1.07e-02	1.81e-02	1.16e-02	0.0135
1/40	3.10e-03	2.05e-03	3.36e-03	2.19e-03	3.78e-03	2.39e-03	0.0148
1/80	7.19e-04	4.70e-04	7.78e-04	5.04e-04	8.76e-04	5.50e-04	0.0322
1/160	1.73e-04	1.13e-04	1.88e-04	1.21e-04	2.11e-04	1.32e-04	0.1560

Table 3. Comparison of pointwise absolute errors of MQ and Spline using $\alpha = 1.75, h = \frac{1}{10}, c = 2.5$, for Example-1

t	Exact	MQ	Absolute Error	[10]
0.1	0.01	0.009999926	7.4031e-08	6.30e-04
0.2	0.04	0.039999854	1.4618e-07	1.29e-03
0.3	0.09	0.089999778	2.2157e-07	1.94e-03
0.4	0.16	0.159999708	2.9205e-07	2.48e-03
0.5	0.25	0.249999642	3.5763e-07	2.88e-03
0.6	0.36	0.359999583	4.1690e-07	3.13e-03
0.7	0.49	0.489999525	4.7452e-07	3.20e-03
0.8	0.64	0.639999473	5.2676e-07	3.12e-03
0.9	0.81	0.809999435	5.6479e-07	2.87e-03
1	1.0	0.999999396	6.0443e-07	2.50e-03

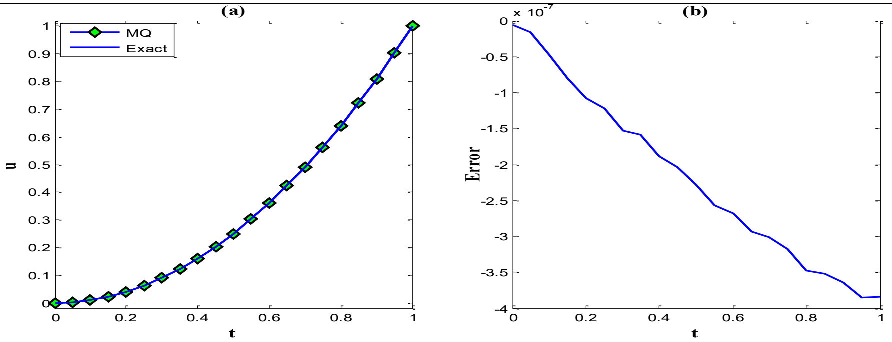


Figure 1. Graphs of (a) MQ and Exact solutions and (b) Error in MQ solution sing $h = {}^{1}/_{20}, \alpha = 1.75,$ for Test Example-1

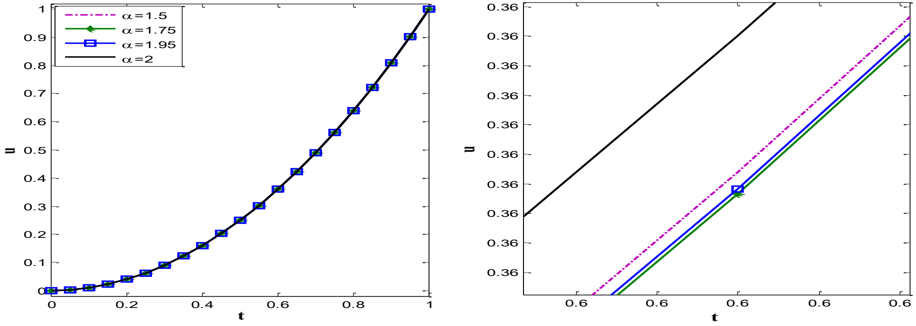


Figure 2. Plots of MQ solutions $\alpha = 1.5, 1.75, 1.95, 2$ using $h = {}^{1}/_{20},$ corresponding to Test Example-1

Test Example-2:

Consider Eqs. (1)-(2) with $A = 1, B = 0, C = 1, g(t) = 0,$ leads to the following FDE [10]:
with $u(0) = 1, u'(0) = 0,$ so that the analytical solution is $u(t) = E_\alpha(-t^\alpha),$ where $E_\alpha(t) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(\alpha j+1)}$.

Table 4-5 reports the $L_\infty$and L_2 errors obtained with MQ and Quintic RBFs for fractional orders $\alpha = 1.5, 1.75, 1.95$ and nodes with step sizes $h = 1/10, 1/20, 1/40, 1/80, 1/160$. The errors decrease consistently as the mesh is refined. For $\alpha = 1.95$ and $h = 1/80$, the MQ method achieves $L_\infty \approx 3.51 \times 10^{-4}$ and $L_2 \approx 2.77 \times 10^{-4}$ while $L_\infty \approx 4.86 \times 10^{-4}$ and $L_2 \approx 3.56 \times 10^{-4}$ for Quintic using $h = 1/160.$

Table 6 presents pointwise absolute errors for $\alpha = 1.8$ with $h = 1/500$ and shape parameter $c = 0.4$. The MQ RBF yields errors on the order of 10^{-4}, which are comparable to those produced in [10].

Figure 3 compares the MQ approximate solution with the exact solution for $\alpha = 1.75$ and $h = 1/20$; the two curves show very close agreement, and the pointwise error remains below 2.5×10^{-4}. Figure 4 illustrates the convergence of the approximate solution as the number of nodes *N* increases for different fractional orders. The plots show that the method achieves stable and consistent convergence.

Table 4. Error Norms of MQ Solutions for Example-2

		α=1.5		α=1.75		α=1.95	
h	c	L_∞	L_2	L_∞	L_2	L_∞	L_2
1/10	7	7.93e-02	4.28e-02	1.39e-02	6.31e-03	1.08e-03	8.37e-04
1/20	6.2	3.72e-02	1.86e-02	7.59e-03	3.21e-03	8.19e-04	6.37e-04
1/40	1.9	1.12.e-02	5.22e-03	3.58e-03	1.51e-03	5.46e-04	4.30e-04

1/80	1.65	4.60e-03	3.40e-03	1.38e-03	7.28e-04	3.51e-04	2.77e-04

Table 5. Error Norms of Quintic Solutions for Example-2

	α=1.5		α=1.75		α=1.95	
h	L_∞	L_2	L_∞	L_2	L_∞	L_2
1/20	3.13e-02	1.52e-02	3.39e-02	2.49e-02	5.27e-02	3.41e-02
1/40	4.32e-02	2.45e-02	2.77e-03	2.05e-03	1.07e-02	6.92e-03
1/80	3.56e-02	2.11e-02	4.59e-03	2.19e-03	6.92e-03	1.57e-03
1/160	2.69e-02	1.63e-02	4.11e-03	2.18e-03	4.86e-04	3.56e-04

Table 6. Comparison of pointwise absolute errors of MQ and Spline using $\alpha = 1.8, h = \frac{1}{500}, c = 0.4$ for Example-2

t	Exact	MQ	Absolute Error	[10]
0.1	0.990565120	0.990678042	1.1292e-04	1.05e-04
0.2	0.967307454	0.967442662	1.3521e-04	8.00e-05
0.3	0.932674161	0.932825580	1.5142e-04	3.10e-05
0.4	0.888098516	0.888223410	1.2489e-04	9.00e-06
0.5	0.834770525	0.834845692	7.5167e-05	2.40e-05
0.6	0.773788499	0.773812801	2.4301e-05	1.30e-05
0.7	0.706211034	0.706185952	2.5082e-05	2.40e-05
0.8	0.633076773	0.632986032	9.0742e-05	7.90e-05
0.9	0.555410776	0.555248551	1.6222e-04	1.44e-04
1	0.474224471	0.473982543	2.4193e-04	2.10e-04

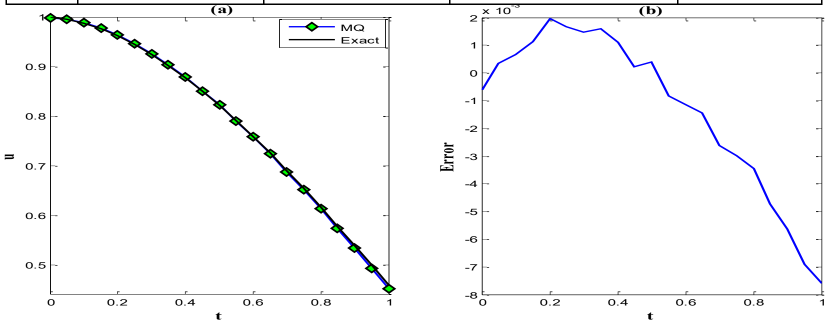


Figure 3. Graphs of (a) MQ and Exact solutions and (b) Error in MQ solution sing $h = {}^{1}/_{20}, \alpha = 1.75,$ for Test Example-2

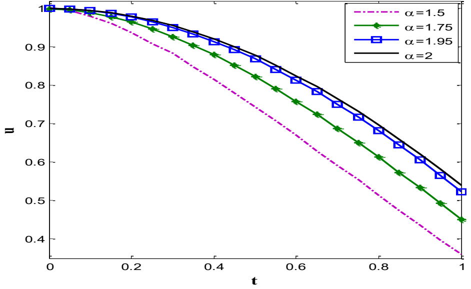


Figure 4. Plots of MQ solutions $\alpha = 1.5, 1.75, 1.95, 2$ using $h = {}^{1}/_{20}$, corresponding to Test Example-2

Test Example-3:

In the third example [11], we take the FDE (1)-(2) with $A = m, B = \mu, C = \kappa, g(t) = \cos(t), u(0) = 0, u'(0) = 1,$ so that Eq. (1) becomes,

$$mD^{\alpha}u(t) + \mu u'(t) + \kappa u(t) = g(t), 1 \leq \alpha \leq 2.$$

This FDE is used to model the displacement $u(t)$ from the equilibrium position in a fractional coupled mass-spring-damper system with mass value m (in kg), with initial displacement $u(0)$ of the given mass and initial velocity $u'(0)$, μ is the damping coefficient (in N-s/m), κ the spring constant (in N/m) and $g(t)$ is an external force. The analytical solution of this problem is not known. We take $m = 1, \mu = 1, \kappa = 1, g(t) = \cos(t), t \in [0,20]$ for simulation.

Example 3 represents a fractional mass-spring-damper system with no known analytical solution. To assess the convergence of the proposed method, the error is estimated using the mesh refinement strategy $E\left(u_h, u_{\frac{h}{2}}\right) = max\left\{\left|u_h(t_i) - u_{\frac{h}{2}}(t_i)\right| : i = 1,2, \dots ,100\right\}$, where u_h denotes the approximate solution obtained with step size h. Table 7 reports these estimated errors for the quintic RBF with $\alpha = 1.9$ as the mesh is successively refined from $h = 0.1$ down to $h = 0.00625$. The estimated error decreases monotonically from 9.16×10^{-3} to 1.61×10^{-3}, indicating consistent convergence as the domain discretization is refined.

Figure 5 displays the approximate solutions for fractional orders $\alpha = 1.9$ and $\alpha = 1.99$ using $h = 1/5$. The solutions exhibit smooth behavior and show a clear dependence on the fractional order; as *a* approaches 2, the solution approaches the integer-order response of the system. The proximity of the two curves for $\alpha = 1.9$ and $\alpha = 1.99$ reflects the continuous dependence of the solution on the fractional order, which is captured accurately by the RBF collocation method.

Table 7. Error in Quintic solution using$\alpha = 1.9$ for Example-3

h	E(u_h,u_(h/2))
0.1	9.1593e-03
0.05	5.5782e-03
0.025	3.2358e-03
0.0125	1.8214e-03
0.00625	1.6103e-03

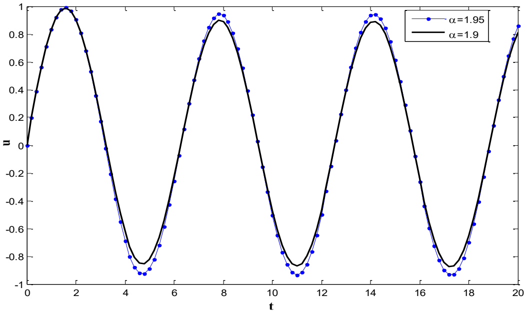


Figure 5. Quintic solutions for $\alpha = 1.9, 1.99, h = \frac{1}{5}$, corresponding to Test Example-3

Conclusion:

This study presents an efficient meshfree radial basis function (RBF) collocation method for solving a class of higher-order fractional differential equations involving the Caputo derivative. The proposed framework transforms the governing equation into a system of linear algebraic equations, improving computational efficiency. Three types of RBFs, including infinitely smooth and piecewise smooth functions, were incorporated into the formulation to assess their performance. The effectiveness of the method was validated through two benchmark problems. A comparative analysis of multiquadric and quintic RBFs indicate that the multiquadric function consistently provides superior accuracy. The performance of the proposed approach was further evaluated using multiple error norms and pointwise absolute error measures. Numerical results confirm that the method achieves high accuracy, with error magnitudes ranging from 10^{-2} down to 10^{-7}, depending on node density and the choice of RBF, while exhibiting strong convergence behavior. Furthermore, comparisons with established numerical techniques demonstrate that the proposed RBF collocation approach attains higher precision for the problems considered. Both graphical and tabulated results confirm the reliability, stability, and consistency of the method. Overall, the findings highlight that the RBF collocation technique is a robust and accurate tool for solving fractional differential equations, particularly those involving memory effects and nonlocal characteristics, which commonly arise in applications such as fluid–structure interaction and anomalous transport phenomena. The meshfree nature of the method also makes it suitable for problems with irregular geometries and scattered data.

Based on the numerical investigations, the multiquadric RBF is recommended when high accuracy is required, whereas piecewise smooth RBFs, such as quintic, provide viable alternatives in situations where shape parameter tuning is challenging. The proposed framework is capable of efficiently solving higher-order fractional differential equations $\alpha \in (1,2]$ with near-exponential convergence behavior. Future work will focus on extending the proposed method to higher-dimensional fractional boundary value problems, incorporating complex boundary conditions, and developing adaptive strategies for optimal node distribution and parameter selection.

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