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ON FOURTH ORDER DIFFERENTIAL EQUATIONS VIA θ-CONTRACTIONS

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In this article, we study the concept of θ-contraction map- ping in rectangular m-metric space. We obtain fixed point result for such mapping which can be approximated by iteration. As an application of the result proved in this paper, the existence of a solution of forth order equations are established. The presented result improves, unify and generalized many known results in the literature.

Introduction

The Banach contraction principle ([6]) is one of the most notable result which has played a vital role in the development of a metric fixed point theory. This principle essentially states that, in a complete metric space (X, d), any contraction $T : X \to X$ that is, if there exists $k \in [0, 1)$

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such that for any $x, y \in X$, we have

(1.0.1) $$d(Tx, Ty) \leq k\, d(x, y),$$

has a unique fixed point. This principle and its variants provide a useful apparatus in guaranteeing the existence and uniqueness of solution of various nonlinear problems: differential equation, integral equation, integro-differential equations. There are many generalizations of the Banach contraction principle in the literature. These generalizations were made either by using the contractive condition or by imposing some additional conditions on an ambient space. There have been a number of generalization of metric space as, rectangular metric space, partial metric space, m-metric space and rectangular m-metric space (see e.g. [1, 2, 4, 11] and references mentioned therein.) Employing one of the above mentioned strategy, Branciari [7], introduced the concept of rectangular metric space and proved an analog of the Banach contraction principle in such spaces.In the recent past, Matthews [9] initiate the concept of partial metric spaces which is the classical extension of a metric space. After that, many researchers generalized some related results in the frame of partial metric spaces. Recently, Asadi et al. [3] introduced the notion of an m-metric space which is the one of interesting generalizations of a partial metric space. Latter on in 2018, Ozgür et al. [10] introduced concept of a rectangular m-metric space and proved analog of the Banach contraction principle on rectangular m-metric spaces. One of the interesting generalizations of Banach contraction principle was given by Jleli et.al in 2014, by introducing a new type of contraction called θ-contraction. Following the authors of [8], let Θ be the set of all functions $\theta: (0, \infty) \longrightarrow (1, \infty)$ satisfying the following conditions:

$\Theta 1$). θ is strictly increasing: $s < t \Rightarrow \theta(s) < \theta(t)$;

$\Theta 2$). For each sequence $\{s_n\}$ in $\mathbb{R}_+$, $\lim_{n\to\infty} s_n = 0$, if and only if $\lim_{n\to\infty} \theta(s_n) = 1$;

$\Theta 3$). There exists $r \in (0, 1)$ and $\ell \in (0, \infty]$ such that $\lim_{s\to 0^+} \frac{\theta(s)-1}{s^r} = \ell$.

Jleli et.al showed that if we take $\theta_B: (0, \infty) \to (1, \infty)$, $\theta_B(s) = e^{\sqrt{s}}$, then $\theta_B \in \Theta$ and the θ_B-contraction reduces to a Banach contraction. Therefore, the Banach contractions are a particular case of θ-contractions. Meanwhile there exist θ -contractions which are not Banach contractions (see [8]). In order to state the main result of [8], we recall the following definition.

Definition 1.0.1. [8] Let (X, d) be a metric space. A mapping $T: X \to X$ is called an θ-contractions if there exists $\theta \in \Theta$ and $k \in (0, 1)$ such that

(1.0.2) $$\theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k$$

holds for all $x, y \in X$ with $d(Tx, Ty) > 0$. The following theorem is the result of Jleli et.al:

Theorem 1.0.2. [8] Let (X, d) be complete generalized metric space and $T: X \to X$ a θ-contractions mapping. Then Thas a unique fixed point.

The aim of this paper, is to prove analog of the $Theorem$ 1.0.2 on rectangular m-metric space. As an application of the result proved in this paper, the existence of a solution of forth order differential equations is established.

PRELIMINARIES

Definition 2.0.3. [10] Let X be a non-empty set. A mapping $m_r : X \times X \to [0,\infty)$ is said to be m_r-metric if for any $x, y \in X$, the following conditions hold:

(1) $m_r(x,y) = m_{r_{x,y}} = M_{r_{x,y}} \Leftrightarrow x = y$,

(2) $M_{r_{x,y}} \leq m_r(x,y)$,

(3) $m_r(x,y) = m_r(y,x)$,

(4) $m_r(x,y) - m_{r_{x,y}} \leq m_r(x,u) - m_{r_{x,u}} + m_r(u,v) - m_{r_{u,v}} + m_r(v,y) - m_{r_{v,y}}$ for all $u, v \in X \setminus \{x,y\}$.

where $m_{r_{x,y}} = \min\{m_r(x,x), m_r(y,y)\}$ and $M_{r_{x,y}} = \min\{m_r(x,x), m_r(y,y)\}$. The pair (X, m_r) is called a rectangular m-metric space.

Definition 2.0.4. [10] A sequence $\{x_n\}$ in a rectangular m-metric space X is said to be: (i) convergent to some $x \in X$ if and only if $\lim_{n\to\infty}(m_r(x_n,x) - m_{r_{x_n,y}} = 0.$ (In this case we write $x_n \to x$ as $n \to \infty$), (ii) a m_r-Cauchy sequence if and only if $\lim_{n,m\to\infty}(m_r(x_n,x_m) - m_{r_{x_n,x_m}})$ and $\lim_{n,m\to\infty}(M_{r_{x_n,x_m}} - m_{r_{x_n,x_m}})$ exist and finite.

A rectangular m-metric space X is said to be m_r-complete if every m_r-Cauchy sequence in X is convergent in X as $\lim_{n,m\to\infty}\left(m_r(x_n,x_m) - m_{r_{x_n,x_m}}\right) = 0$ and $\lim_{n,m\to\infty}\left(M_{r_{x_n,x_m}} - m_{r_{x_n,x_m}}\right) = 0.$

Lemma 2.0.5. [10] Assume that $x_n \to x$ and $y_n \to y$ as $n \to \infty$ in a rectangular m-metric space. Then $\lim_{n\to\infty}\left(m_r(x_n,y_n) - m_{r_{x_n,y_n}}\right) = m_r(x,y) - m_{r_{x,y}}$.

Lemma 2.0.6. [10] Assume that xn $\to$ x as n $\to \infty$ in a rectangular m-metric space. Then

$$\lim_{n\to\infty}\left(m_r(x_n,y) - m_{r_{x_n,y}}\right) = m_r(x,y) - m_{r_{x,y}}, \qquad \forall\, y \in X.$$

MAIN RESULT

Definition 3.0.7. Let (X, m_r) be a rectangular m-metric space and $\theta \in \Theta$. A mapping $T : X \to X$ is called θ-contraction if there exist $k \in (0,1)$ such that for all $x, y \in X$ with $m_r(Tx,Ty) > 0$ we have

$$\theta(m_r(Tx,Ty)) \leq [\theta(m_r(x,y))]^k\,. \tag{3.0.3}$$

Before stating the main result, we first prove following lemma for the class of θ-contraction mappings for rectangular m-metric space.

Lemma 3.0.8. Let T be a θ-contraction on rectangular m-metric space (X, m_r). If Picard iteration defined by

$$x_m = Tx_{m-1}, \quad m \in N, \tag{3.0.4}$$

where $x_0 \in X$. converges to $\in X$, then $\lim_{n\to\infty} Tx_m = Tu^*$.

Proof. We divide the proof into two cases.

Case 1 : Suppose that

(3.0.5) $$\lim_{n\to\infty} m_r(Tx_m, Tu^*) = 0.$$

Since, $m_{r_{T x_m, Tu^*}} = \min\{m_r(Tx_m, Tx_m), m_r(Tu^*, Tu^*)\} \leq m_r(Tx_m, Tu^*)$. Letting $m \to \infty$, into the inequality, we have $\lim_{m\to\infty} m_{r_{T x_m, Tu^*}} \leq \lim_{m\to\infty} m_r(Tx_m, Tu^*)$. By using (3.0.5), it follows that

(3.0.6) $$\lim_{m\to\infty} m_r(Tx_m, Tu^*) = 0.$$

From (3.0.5) and (3.0.6), we get $\lim_{m\to\infty} (m_r(T x_m, Tu^*) - m_{r_{T x_m, Tu^*}}) = 0$.

Hence, $Tx_m \to Tu^*$ as $m \to \infty$.

Case 2 : On the other hand, suppose that

(3.0.7) $$\lim_{m\to\infty} m_r(Tx_m, Tu^*) > 0$$

Since $m_r(Tx_m, Tu^*) \in [0, \infty)$, for all $m \in \mathbb{N}$ Therefore, there exists $N \in \mathbb{N}$ such that

(3.0.8) $$m_r(Tx_m, Tu^*) > 0, \forall m \geq N.$$

By using (3.0.3) and (3.0.8), we obtain that

$$\theta(m_r(Tx_m, Tu^*)) \leq [\theta(m_r(x_m, u^*))]^k < \theta(m_r(x_m, u^*)), \quad \forall m \geq N.$$

By using ($\Theta 1$), we get

(3.0.9) $$m_r(Tx_m, Tu^*) < m_r(x_m, u^*), \quad \forall m \geq N.$$

Case (a) : Suppose that

(3.0.10) $$m_r(u^*, u^*) \leq \lim_{m\to\infty} m_r(x_m, x_m).$$

In this case, our claim is to show that $m_r(u^*, u^*) = 0$. If $\lim_{m\to\infty} m_r(x_m, x_m) = 0$. Then it follows from (3.0.10), we have $m_r(u^*, u^*) = 0$. On the other hand, if $\lim_{m\to\infty} m_r(x_m, x_m) > 0$. Therefore, there exists $M \in \mathbb{N}$ such that

(3.0.11) $$m_r(x_m, x_m) > 0, \quad \forall m \geq M.$$

By using (3.0.3), we have $\theta(m_r(x_m, x_m)) \leq [\theta(m_r(x_{m-1}, x_{m-1}))]^k$, $\forall m \geq M$. Continuing this way, we can obtain $\theta(m_r(x_m, x_m)) \leq [\theta(m_r(x_0, x_0))]^{k^m}$, $\forall m \geq M$. It follows that

(3.0.12) $$\lim_{m\to\infty} \theta(m_r(x_m, x_m)) = 1.$$

By using ($\Theta 2$) in (3.0.12), we get $\lim_{m\to\infty} m_r(x_m, x_m) = 0$. By using (3.0.10), we obtain $m_r(u^*, u^*) \leq \lim_{m\to\infty} m_r(x_m, x_m) = 0$. It follows that

(3.0.13) $$m_r(u^*, u^*) = 0.$$

Since $m_r(Tu^*, Tu^*) \in [0, \infty)$. If $m_r(Tu^*, Tu^*) = 0$. Then from (3.0.13), we have

(3.0.14) $$m_r(Tu^*, Tu^*) = 0 = m_r(u^*, u^*).$$

On the other hand, if $m_r(Tu^*, Tu^*) > 0$. It follows from (3.0.3), we get

$$\theta(m_r(Tu^*, Tu^*)) \leq [\theta(m_r(u^*, u^*))]^k < \theta(m_r(u^*, u^*)).$$

By using ($\Theta1$), we obtain $m_r(Tu^*, Tu^*)) < m_r(u^*, u^*)$. By using (3.0.13), the above inequality becomes

(3.0.15) $$m_r(Tu^*, Tu^*)) = 0 = m_r(u^*, u^*).$$

In this case, from (3.0.14) and (3.0.15), we conclude that

(3.0.16) $$m_{r_{x_m,u^*}} = min\{m_r(x_m, x_m), m_r(u^*, u^*)\} = m_r(u^*, u^*) = 0, \forall m \geq M.$$

(3.0.17) $$m_{r_{Tx_m,Tu^*}} = \min\{m_r(Tx_m, Tx_m), m_r(Tu^*, Tu^*)\} = m_r(Tu^*, Tu^*) = 0, \forall m \geq M.$$

Since, $x_m \to u^*$ as $m \to \infty$. This implies that $m_r(x_m, u^*) - m_{r_{x_m,u^*}} \to 0$, $m \to \infty$. By using (3.0.16), into the above inequality, we have

(3.0.18) $$m_r(x_m, u^*) \to 0, \quad m \to \infty.$$

Letting m $\to \infty$, in (3.0.9), we have

(3.0.19) $$\lim_{m\to\infty} m_r(Tx_m, Tu^*) < lim\, m \to \infty \lim_{m\to\infty} m_r(x_m, u^*)$$

By using (3.0.18) into the (3.0.19), we get

(3.0.20) $$\lim_{m\to\infty} m_r(Tx_m, Tu^*) = 0.$$

Letting $m \to \infty$, in (3.0.17), we have

(3.0.21) $$m_{r_{Tx_m,Tu^*}} \to 0, \quad m \to \infty$$

By combining (3.0.20) and (3.0.21), we obtain $m_r(Tx_m, Tu^*) - m_{r_{Tx_m,Tu^*}} \to 0$, $m \to \infty$. Thus $Txm \to Tu^*\, as\, m \to \infty$.

Case (b) : On the other hand, suppose that

(3.0.22) $$m_r(u^*, u^*) \geq \lim_{m\to\infty} m_r(x_m, x_m).$$

Then in this case, our claim is to show that

(3.0.23) $$\lim_{m\to\infty} m_{r_{Tx_m,Tu^*}} = 0.$$

If $m_r(u^*, u^*) = 0$. Then it follows form (3.0.22), we obtain (3.0.23). If $m_r(u^*, u^*) > 0$. Then in this case, we further assume that $\lim_{m\to\infty} m_r(x_m, x_m) = 0$. Then it is easy to see that $\lim_{m\to\infty} m_{r_{x_m,u^*}} = 0$. Now on the other hand, suppose that $\lim_{m\to\infty} m_r(x_m, x_m) > 0$. Based on the same procedure as in case (a), we obtain $\lim_{m\to\infty} m_r(x_m, x_m) = 0$. Hence, our claim is true. As $x_m \to u^*$ as $m \to \infty$. We have $m_r(x_m, u^*) - m_{r_{x_m,u^*}} \to 0$, as $m \to \infty$. By using (3.0.23), into the above, it follows that

(3.0.24) $$\lim_{m\to\infty} m_r(x_m, u^*) = 0.$$

By using (3.0.24), it follows from (3.0.9), we have the following

(3.0.25) $$\lim_{m\to\infty} m_r(Tx_m, Tu^*) = 0.$$

Moreover, we have

(3.0.26) $$m_{r_{Tx_m,Tu^*}} \le m_r(Tx_m, Tu^*).$$

Letting m → ∞, we obtain

(3.0.27) $$\lim_{m\to\infty} m_{r_{Tx_m,Tu^*}} = 0.$$

By using (3.0.26) and (3.0.27), we have $Tx_m \to Tu^*$ as $m \to \infty$.

Proposition 3.0.9. Let (X, m_r) be a rectangular m-metric space and $T : X \to X$ be a θ-contraction mapping. If the Picard iteration defined by $x_m = Tx_{m-1}$ for $m \ge 1, (x_0 \in X)$ has the property $m_r\,(x_n, x_n) = 0$ for some $n \in \mathbb{N}$. Then

(3.0.28) $$m_r(x_m, x_m) = 0, \ \forall m \ge n.$$

Proof. We will prove it by induction on m. Suppose that the result is true for $m = k > n$. This can also be expressed as

(3.0.29) $$m_r(x_k, x_k) = 0.$$

We want to prove that $m_r(x_{k+1}, x_{k+1}) = 0$. On contrary suppose that $m_r(x_{k+1}, x_{k+1}) > 0$. By using (3.0.3), we have $\theta\,(m_r(x_{k+1}, x_{k+1})) \le [\theta\,(m_r(x_k, x_k))]^k < \theta(m_r(x_k, x_k))$. It follows from (Θ1),$m_r(x_{k+1}, x_{k+1}) < m_r(x_k, x_k)$.By using (3.0.29) into the above inequality, we obtain the desired result.

Proposition 3.0.10. Let (X, m_r) be a rectangular m-metric space and $T : X \to X$ be a θ-contraction mapping. Suppose that the Picard iteration defined by $x_m = Tx_{m-1}$ $m \ge 1, (x_0 \in X)$, Then, for every fixed $n \in \mathbb{N}$, we have

(3.0.30) $$m_{r_{x_n,x_m}} = min\{m_r(x_n, x_n), m_r(x_m, x_m)\} = m_r(x_m, x_m), \quad m > n.$$

Proof. On contrary suppose that

(3.0.31) $$m_{r_{x_n,x_m}} = m_r(x_n, x_n), \quad \forall m > n.$$

Now, we divide the proof into two following cases.

Case 1 : If $m_r(x_n, x_n) = 0$. By *Proposition* 3.0.9, we have$m_r(x_m, x_m) = 0$, ∀m > n.

Consider $m_{r_{x_n,x_m}} = min\{m_r(x_n, x_n), m_r(x_m, x_m)\} = min\{0,0\} = 0 = m_r(x_m, x_m), \forall m > n$. Therefore, it follows that $m_{r_{x_n,x_m}} = m_r(x_m, x_m), \forall m > n$.

Case 2 : If $m_r(x_n, x_n) > 0$. It follows from (3.0.31),

(3.0.32) $$m_r(x_m, x_m) > 0 \ \forall m > n.$$

By using (3.0.3) and (3.0.32), we have

$$\theta\left(m_r(x_m, x_m)\right) \le \left[\theta\left(m_r(x_m, x_m)\right)\right]^k \le \cdots \le \left[\theta\left(m_r(x_n, x_n)\right)\right]^{k^{m-n}} < \theta\,(m_r(x_n, x_n)),$$

By using ($\Theta1$), we obtain $m_r(x_m, x_m) < m_r(x_n, x_n)$; $\forall m > n$, which is a contradiction on the fact that (3.0.31).

Theorem 3.0.11. Let (X, m_r) be a complete rectangular m-metric space and $T : X \to X$ be a θ-contraction mapping. Then T is a Picard operator.

Proof. We divide the proof into two following cases.

Case 1 : If there exists a natural number n such that $x_{n+1} = x_n$. Clearly, in this case x_n is a fixed point of T.

Case 2 : Now suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. We divide this case into two further subcases. Subcase 1 : If

(3.0.33) $$m_r(x_{n+1}, x_n) = 0,$$

for some $n \in \mathbb{N}$. Notice that $m_{r_{x_{n+1},x_n}} \leq m_r(x_{n+1}, x_n) = 0$. Therefore, it follows that

(3.0.34) $$m_{r_{x_{n+1},x_n}} = 0.$$

Furthermore, it follows from the *Proposition* 3.0.10, we have

(3.0.35) $$m_{r_{x_{n+1},x_n}} = m_r(x_{n+1}, x_{n+1}).$$

By combining (3.0.34) and (3.0.35), we get

(3.0.36) $$m_r(x_{n+1}, x_{n+1}) = 0.$$

Since $m_r(x_{n+1}, x_{n+1}) = 0.$, it follows from the *Proposition* 3.0.9, we obtain

(3.0.37) $$m_r(x_{n+2}, x_{n+2}) = 0.$$

Now from here, we further divide subcase 1, into two cases.

Subcase 1_a: If

(3.0.38) $$m_r(x_{n+1}, x_{n+2}) = 0.$$

It follows from (3.0.36), (3.0.37) and (3.0.38), we have

$$m_r(x_{n+1}, x_{n+1}) = m_r(x_{n+2}, x_{n+2}) = m_r(x_{n+1}, x_{n+2})\ 0.$$

By using the property of m_r, it follows that $x_{n+1} = x_{n+2}$. Moreover, this equation can also be written as $x_{n+1} = Tx_{n+1}$. Clearly, x_{n+1} is the fixed point. As a result, the theorem is proved.

Subcase1_b : On the other hand, suppose that

(3.0.39) $$m_r(x_{n+1}, x_{n+2}) > 0.$$

By using (3.0.3), we have

$$\theta\,(m_r(x_{n+1}, x_{n+2})) \leq [\theta\,(m_r(x_n, x_{n+1}))]^k < \theta\,(m_r(x_n, x_{n+1})).$$

By using (Θ1), this gives $m_r(x_{n+1}, x_{n+2}) < m_r(x_n, x_{n+1})$. By using (3.0.33) we have $m_r(x_{n+1}, x_{n+2}) = 0$, which is a contradiction of the fact (3.0.39).

Subcase 2 : Now, we suppose that $m_r(x_{n+1}, x_n) > 0$, for all $n \in \mathbb{N}$. Let $\beta_n = m_r(x_n, x_{n+1}) \forall n \in N$. Then by (3.0.3), we get

(3.0.40) $$\theta\,(\beta_n) \leq [\theta\,(\beta_{n-1})]^k \leq [\theta\,(\beta_{n-2})]^{k^2} \leq \cdots \leq [\theta\,(\beta_0)]^{k^n}, \forall\, n \in \mathbb{N}.$$

Taking letting $n \to \infty$ in (3.0.40), we get $\lim_{n\to\infty} \theta(\beta_n) = 1$. By using $(\Theta 2)$, we have $\lim_{n\to\infty} \beta_0 = 0$. Now from $(\Theta 3)$, there exists $r \in (0,1)$ and $\ell \in (0,\infty]$ such that $\lim_{n\to\infty} \theta(\beta_n) - 1/_{r} = \ell$.

Suppose that $\ell < \infty$. In this case, let

(3.0.41) $$B = \ell/2 > 0.$$

It follows that, there exists $n_0 \in \mathbb{N}$ such that $\left|\frac{\theta(\beta_n)-1}{(\beta_n)^r} - \ell\right| \leq B,\ \forall n \geq n_0$. It follows that $-B + \ell \leq \frac{\theta(\beta_n)-1}{(\beta_n)^r}$, $\forall n \geq n_0$. By using (3.0.41), we obtain

$$B \leq \frac{\theta(\beta_n)-1}{(\beta_n)^r}, \qquad \forall n \geq n_0.$$

It asserts that $n(\beta_n)^r \leq An[\theta(\beta_n) - 1], \forall n \geq n_0$, where $A = 1/B$.

Suppose now that $\ell = \infty$. Let $B > 0$ be an arbitrary positive number. It follows that, there exists $n_0 \in \mathbb{N}$. such that

$$\frac{\theta(\beta_n)-1}{(\beta_n)^r} \geq B, \qquad \forall n \geq n_0.$$

By following the above process, we have $n(\beta_n)^r \leq An[\theta(\beta_n) - 1], \forall n \geq n_0$, where $A = 1/B$. Therefore if $\ell \in (0,\infty]$, there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that

$$n(\beta_n)^r \leq An[\theta(\beta_n) - 1], \qquad \forall n \geq n_0.$$

By using (3.0.3), we obtain $n(\beta_n)^r \leq An\left[\theta(\beta_0)^{k^n} - 1\right], \quad \forall n \geq n_0$.

Letting $n \to \infty$ in the inequality, we obtain $\lim_{n\to\infty} n(\beta_n)^r = 0$. Thus, there exist $n_1 \in \mathbb{N}$ such that $\beta_n = m_r(x_n, x_{n+1}) \leq \frac{1}{n^{1/r}} 1, \qquad \forall n \geq n_1$.

Now, we will prove that

(3.0.42) $$\lim_{n\to\infty} m_r(x_n, x_{n+2}) = 0.$$

If $m_r(x_n, x_{n+2}) = 0$ for all $n \in \mathbb{N}$, then we have (3.0.42). On the other hand, if $m_r(x_n, x_{n+2}) > 0$ for all $n \in \mathbb{N}$. By using (3.0.3), we get

$$\theta\left(m_r(x_n, x_{n+2})\right) \leq \left[\theta\left(m_r(x_{n-1}, x_{n+1})\right)\right]^k, \qquad \forall n \in \mathbb{N}.$$

Continuing this way, we have

(3.0.43) $$\theta\left(m_r(x_n, x_{n+2})\right) \leq \left[\theta\left(m_r(x_0, x_2)\right)\right]^{k^n}, \quad \forall n \in \mathbb{N}.$$

Letting $n \to \infty$ in (3.0.43), we obtain $\lim_{n\to\infty} \theta\left(m_r(x_n, x_{n+2})\right) = 1$. By using $(\Theta 2)$, we get $\lim_{n\to\infty} m_r(x_n, x_{n+2}) = 0$. Now, we prove that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is a m_r-Cauchy.

Let $m > n$ with $m = n + o$ where $o > 2$, we will consider two cases.

Case (i): For o is odd. Let $o = 2p + 1$, where $p \in \mathbb{N}$. Then $m_r(x_n, x_m) - m_{r_{x_n,x_m}} = m_r(x_n, x_{n+2p+1}) - m_{r_{x_n,x_{n+2p+1}}}$

$$\leq m_r(x_n, x_{n+1}) - m_{r_{x_n,x_{n+1}}} + \cdots + m_r(x_{n+2p}, x_{n+2p+1}) - m_{r_{x_{n+2p},x_{n+2p+1}}}$$

$$< m_r(x_n, x_{n+1}) + \cdots + m_r(x_{n+2p}, x_{n+2p+1}) = \beta_n + \cdots + \beta_{n+2p}$$

$$\leq \sum_{n=1}^{\infty} \beta_i \leq \sum_{n=1}^{\infty} 1/_{i^{1/r}} < \epsilon.$$

Case (ii): For o is even. Let $o = 2p$, where $p \in \mathbb{N}$ Then

$$m_r(x_n, x_m) - m_{r_{x_n,x_m}} = m_r(x_n, x_{n+2p+1}) - m_{r_{x_n,x_{n+2p+1}}}$$

$$\leq m_r(x_n, x_{n+2}) - m_{r_{x_n,x_{n+2}}} + m_r(x_{n+2}, x_{n+3}) - m_{r_{x_{n+2},x_{n+3}}} + \cdots + m_r(x_{n+2p-1}, x_{n+2p}) - m_{r_{x_{n+2p-1},x_{n+2p}}}$$

$$< m_r(x_n, x_{n+2}) + m_r(x_{n+2}, x_{n+3}) + \cdots + m_r(x_{n+2p-1}, x_{n+2p})$$

$$\leq m_r(x_n, x_{n+2}) + \sum_{n=1}^{\infty} \beta_i \leq m_r(x_n, x_{n+2}) + \sum_{n=1}^{\infty} 1/_{i^{1/r}} < \epsilon.$$

Indeed, the series $\sum_{n=1}^{\infty} 1/_{i^{1/r}}$ converges and $\lim_{n\to\infty} m_r(x_n, x_{n+2}) = 0$, this implies that $\lim_{n,m\to\infty} (m_r(x_n, x_m) - m_{r_{x_n,x_m}})$, exist and finite.

Now, if $M_{r_{x_n,x_m}} = 0$ for all $m > n$, then $m_{r_{x_n,x_m}} = 0$ for all $m > n$, which implies that $M_{r_{x_n,x_m}} - m_{r_{x_n,x_m}} = 0, \forall m > n$. This implies that $\lim_{n,m\to\infty} (M_r(x_n, x_m) - m_{r_{x_n,x_m}}) = 0$.

Now, we may assume that $M_{r_{x_n,x_m}} > 0, \forall m > n$. From *Proposition* 3.0.10, we obtain $M_{r_{x_n,x_m}} = m_r(x_n, x_n) > 0, \forall m > n$. Suppose $\mu_n = m_r(x_n, x_n)$ for all $n \in \mathbb{N}$. Then by (3.0.3), we obtai

(3.0.44) $$\theta(\mu_n) \leq [\theta(\mu_{n-1})]^k \leq [\theta(\mu_{n-2})]^{k^2} \leq \cdots \leq [\theta(\mu_0)]^{k^n}.$$

On taking limit as $n \to \infty$, in (3.0.44), we get $\lim_{n\to\infty} \theta(\mu_n) = 1$. By using $(\Theta 2)$, we have

(3.0.45) $$\lim_{n\to\infty} \mu_n = 0.$$

Based on the same procedure, we obtain $\mu_n = m_r(x_n, x_n) \leq 1/_{n^{1/r_1}}, \forall n \geq n_3$. Therefore, we obtain

$$M_{r_{x_n,x_m}} - m_{r_{x_n,x_m}} = m_r(x_n, x_n) - m_r(x_m, x_m)$$

$$< m_r(x_n, x_n) + m_r(x_{n+1}, x_{n+1}) + \cdots + m_r(x_m, x_m)$$

$$\leq \mu_n + \mu_{n+1} + \cdots + \mu_m \leq \sum_{n=1}^{\infty} \mu_i \leq \sum_{n=1}^{\infty} 1/_{n^{1/r_1}} < \epsilon.$$

Since the series $\sum_{n=1}^{\infty} {}^{1}/_{n^{1/r_1}}$ converges, this implies that $\lim_{n,m\to\infty}(m_r(x_n,x_m) - m_{r_{x_n,x_m}})$ exist and finite. Based on the above argument we conclude that $\{x_n\}_{n\in\mathbb{N}}$ is an m_r-Cauchy sequence. Since (X,m_r) is a complete rectangular m-metric space so x_n converges to $u^* \in X$. Since $m_r(x_n,x_n) > 0$, by using $(3.0.3)$ and $(\Theta 2)$, we conclude that

(3.0.46) $$\lim_{n\to\infty} m_r(x_n,Tx_n) = 0.$$

By using $m_{r_{x_n,Tx_n}} \le m_r(x_n,Tx_n)$, we have

(3.0.47) $$\lim_{n\to\infty} m_{r_{x_n,Tx_n}} = 0.$$

By using $(3.0.46)$ and $(3.0.47)$, we have

(3.0.48) $$\lim_{n\to\infty} m_r(x_n,Tx_n) - m_{r_{x_n,Tx_n}} = 0.$$

Since

(3.0.49) $$x_n \to u^*, \quad n \to \infty.$$

Therefore, it follows from *Lemma* 3.0.8, we obtain

(3.0.50) $$Tx_n \to Tu^*, \quad n \to \infty.$$

By using $(3.0.49)$ and $(3.0.50)$ into the *Lemma* 2.0.5, then $(3.0.48)$ becomes

(3.0.51) $$m_r(u^*,Tu^*) = m_{r_{u^*,Tu^*}}.$$

By *Proposition* 3.0.10, we have $m_{r_{x_n,Tx_n}} = m_r(Tx_n,Tx_n), \forall n \in \mathbb{N}$. Letting $n \to \infty$ in the this equality, we obtain $\lim_{n\to\infty} m_{r_{x_n,Tx_n}} = \lim_{n\to\infty} m_r(Tx_n,Tx_n)$ Therefore, we have

(3.0.52) $$\lim_{n\to\infty}(m_{r_{x_n,Tx_n}} - m_r(Tx_n,Tx_n)) = 0.$$

By using $(3.0.49)$ and $(3.0.50)$ into the *Lemma* 2.0.5, then $(3.0.52)$ becomes

(3.0.53) $$m_{r_{u^*,Tu^*}} - m_r(Tu^*,Tu^*) = 0.$$

By using $(3.0.51)$ and $(3.0.53)$, we have

(3.0.54) $$m_r(u^*,Tu^*) = m_{r_{u^*,Tu^*}} = m_r(Tu^*,Tu^*).$$

From $(3.0.48)$, we obtain

(3.0.55) $$\lim_{n\to\infty}(m_r(x_n,x_{n-1}) - m_{r_{x_n,Tx_n}}) = 0.$$

By using $(3.0.49)$ and $(3.0.50)$ into the *Lemma* 2.0.5, then $(3.0.55)$ becomes

(3.0.56) $$m_r(u^*,u^*) = m_{r_{u^*,Tu^*}}.$$

By using $(3.0.54)$ and $(3.0.56)$, we get $m_r(u^*,u^*) = m_r(Tu^*,u^*) = m_r(Tu^*,Tu^*)$. This implies that $Tu^* = u^*$.

For the uniqueness of fixed point, suppose that there exist two elements $x,y \in X$ such that $x = T\,x$ and $y = T\,y$ with $x \neq y$. In order to proof the uniqueness of the fixed point, we divide into two cases.

Case A : If $m_r(T\,x, T\,y) = m_r(x,y) = 0$. Without loss of generality, suppose that $m_{r_{x,y}} = m_r(x,x)$. Notice that $m_r(x,x) = m_{r_{x,y}} \le m_r(x,y) = 0$. It follows that $m_r(x,x) = 0$. Further, we divide case A into two subcases.

Subcase A1 : If $m_r(y,y) = 0$. Then it is easy to check that $x = y$.

Subcase A2 : On the other hand, if $m_r(y,y) > 0$. By using $(3.0.3)$, we have $(m_r(y,y)) < \theta(m_r(y,y))$,. By $(\Theta 1)$, we get $m_r(y,y) < m_r(y,y)$. which is a contradiction.

Case B : If $m_r(T\,y, T\,y) = m_r(x,y) > 0$. By using $(3.0.3)$, we deduce $\theta(m_r(x,y)) < \theta(m_r(x,y))$. By using $(\Theta 1)$, we have $m_r(x,y) < m_r(x,y)$. which leads to a contradiction.

We now provide examples that support $Theorem$ $3.0.11$.

Example 3.1. Let $X = [0,\infty)$ be endowed with rectangular m-metric $m_r(x,y) = \frac{|x|+|y|}{2}$ for all $x, y \in X$. Then (X, m_r) is an complete rectangular m-metric space. Let $\theta : (0,\infty) \to (1,\infty)$ be a mapping defined as $\theta(x) = e^{\sqrt{x}}$, for all $x \in X$. Define $T : X \to X$ as $(x) = \frac{x}{2}$, for all $x \in X$. It is easy to check that for the value of $k = \sqrt{1/2}$, T is θ-contraction. Therefore, by $Theorem$ $3.0.11$, T has a unique fixed point 0.

Example 3.2. Let $X = \{1,2,3,4\}$. Define $m_r : X \times X \to [0,\infty)$ as

$$m_r(1,1) = m_r(2,2) = m_r(3,3) = 0 \; and \; m_r(4,4) = 5$$
$$m_r(1,2) = m_r(2,1) = 3, \quad m_r(1,3) = m_r(3,1) = 1, m_r(1,4) = m_r(4,1) = 4$$
$$m_r(2,3) = m_r(3,2) = 1, \quad m_r(2,4) = m_r(4,2) = 4, \quad m_r(3,4) = m_r(4,3) = 4.$$

Clearly, (X, m_r) is a complete rectangular m-metric space. On the other hand, the (X, m_r) is not a m-metric space.Define $T : X \to X$ as

$$T(x) = \begin{cases} 1, & x = 1,2,3 \\ 3, & x = 4 \end{cases}$$

For $x \in \{1,2,3\}$ and $y = 4$, we have $m_r(T\,x, T\,y) = m_r(1,3) = 1 > 0$. Therefore,

$$e^{\sqrt{m_r(T(x),T(y))\, e^{m_r(T(x),T(y))}}} \le \left[e^{\sqrt{m_r(x,y) e^{m_r(x,y)}}}\right]^{0.9}.$$

Suppose $\theta(x) = e^{\sqrt{x e^x}}$ and $k = 0.9$, we see that T is a θ-contraction which satisfies $Theorem$ $3.0.11$. Moreover, $x = 1$ is the fixed point of T.

APPLICATION TO FOURTH ORDER DIFFERENTIAL EQUATION

In this section, we apply $Theorem$ $3.0.11$ to find the existence and uniqueness of fourth order differential equation. We consider the problem

$$(4.0.57) \qquad \begin{cases} y^4(t) = g(t, y(t), y', y'', y'''), \\ y(0) = y'(0) = y''(1) = y'''(1) = 0; \quad t \in [0,1], \end{cases}$$

where $g: [0,1] \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is a continuous function. This problem known as a boundary value problem is employed to model such phenomena as deformations of an elastic beam in its equilibrium state, where one endpoint is free while the other is fixed. In this section, we study the existence of solution of a fourth order differential equation boundary value problem. Let $S = C[0,1]$, where $C[0,1]$ represents the space of all continuous functions defined on the closed interval [0, 1]. A rectangular m-metric space on S (see [5]) is given by

$$m_r(x,y) = \sup_{t\in[0,1]} \frac{|x(t)+y(t)|}{2}.$$

Note that the space $S = (C[0,1], m_r)$ is a complete rectangular m-metric space. It follows from [1] that the boundary value problem for (4.0.57) can be written in the following integral form:

$$y(t) = \int_0^1 \mathcal{G}(t,s) g\big(s, y(s), y'(s)\big) ds, \quad y \in C[0,1],$$

where $\mathcal{G}(t,s)$ is Green's function of the homogenous linear problem $y^4(t) = y(0) = y'(0) = y''(1) = y'''(1) = 0$, which is explicitly given by

(4.0.58) $$\mathcal{G}(t,s) = \begin{cases} \frac{1}{6} t^2(3s-t), & 0 \le t \le s \le 1 \\ \frac{1}{6} s^2(3t-s), & 0 \le s \le t \le 1 \end{cases}$$

Theorem 4.0.12. Assume that the following conditions are satisfied:

1) $g : [0,1] \times R\,3 \times R \to R$ is continuous.
2) There exists $\tau \in [1,\infty)$ such that the following condition holds for all x, y ∈ S
$$\big|g(s, x(s), x'(s))\big| + \big|g(s, y(s), y'(s))\big| \le 6e^{-\tau}(|x(s)| + |y(s)|), \quad s \in [0,1]$$
3) There exists $y_0 \in X$ such that, for all $t \in [0,1]$, we have
$$y_0 t = \int_0^1 \mathcal{G}(t,s) g\big(s, y_0(s), y_0'(s)\big) ds.$$

Then the boundary value problem (4.0.57) has a solution in S.

Proof. If we define a mapping $T: S \to S$ by

$$T(y)(t) = \int_0^1 \mathcal{G}(t,s) g\big(s, y(s), y'^{(s)}\big) ds,$$

then $y = T(y)$, which yields that boundary value problem has a unique solution. Consider

$$\frac{|T(x)(t)| + |T(y)(t)|}{2} = \frac{\left|\int_0^1 \mathcal{G}(t,s) g(s, x(s), x'(s)) ds\right| + \left|\int_0^1 \mathcal{G}(t,s) g(s, y(s), y'(s)) ds\right|}{2}$$
$$\le \int_0^1 \mathcal{G}(t,s) \left(\frac{|g(s,x(s),x'(s))| + |g(s,y(s),y'(s))|}{2}\right) ds$$

$$\leq \int_0^1 \frac{1}{2} s^2\, 6e^{-\tau}\, \frac{|x(s)| + |y(s)|}{2} ds$$

$$\leq 6e^{-\tau}\, m_r(x,y) \int_0^1 \frac{1}{2} s^2\, ds$$

$$= e^{-\tau}\, m_r(x,y),$$

which yields

(4.0.59) $$m_r\big(T(x),T(y)\big) \leq e^{-\tau}\, m_r(x,y)$$

where $0 < e^{-\tau} < 1$ as $-\tau \geq 1$. So we can say

(4.0.60) $$e^{m_r(T(x),T(y))} \leq e^{m_r(x,y)}.$$

By using (4.0.59) and (4.0.60), we have

$$m_r\big(T(x),T(y)\big)e^{m_r(T(x),T(y))} \leq e^{-\tau}\, m_r(x,y)e^{m_r(x,y)}.$$

Further, we have

$$e^{\sqrt{m_r(T(x),T(y))e^{m_r(T(x),T(y))}}} \leq \left[e^{\sqrt{m_r(x,y)e^{m_r(x,y)}}}\right]^{\sqrt{e^{-\tau}}}$$

where, $r = \sqrt{e^{-\tau}}$ hence

$$\theta\big(m_r(T(x),T(y))\big) \leq \big[\theta\big(m_r(x,y)\big)\big]^r$$

which gives $\theta(x) = e^{\sqrt{xe^x}}$. So conditions of $Theorem$ 3.0.11 are satisfied. Hence, the fourth order differential equation given in (4.0.57) has a unique solution.

CONCLUSION

(1) We introduced the class of θ-contraction mappings in the set up of rectangular m-metric spaces.
(2) We obtain fixed point $Theorem$ 3.0.11 in the context of rectangular m-metric spaces and justify our result with some examples.
(3) As an application of our result ($Theorem$ 3.0.11), the existence of the solution to the problem of forth order differential equation is presented.

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