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# COMMON FIXED POINTS OF CONTRACTIVE MAPPINGS IN b-METRIC-LIKE SPACES

Original Article

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In this paper we discuss common fixed point result of b-metric-like space. As an application, we prove certain common fixed-point results in the setup of such spaces for $\beta$-ordered contractive mapping. Finally, one example is presented in order to verify the effectiveness and applicability of our main results.

## Introduction

There are many generalizations of the notion of metric space. Alber and Guerre-Delabrere [8] introduced the theme of a weakly contractive mapping defined on a Hilbert space is a Picard operator. Rhoades [32] proved the same results considering a complete metric space as an

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alternative of Hilbert space as a domain of the mapping.
Czerwik [17] and Matthews [27] introduced the concepts of $b$-metric space and partial metric space respectively. On the basis of these two concepts, Shukla [34] proved another idea which is called a partial $b$-metric space. Since then, several papers have been published on the fixed point theory of various classes of single-valued and multi-valued operators in $b$-metric spaces (see also [12,13,35]). Pacurar [29] given some results on sequences of almost contractions and fixed points in $b$-metric spaces. Hussain and Shah [22] obtained few results on KKM mappings in cone $b$-metric spaces. Recently, Khamsi [25] and Khamsi and Hussain [26] have dealt with spaces of this kind, although under different names (in the spaces called metric-type) and obtained (common) fixed point results. In particular, they showed that most of the new fixed point existence results of contractive mappings defined on such metric spaces are merely copies of the classical ones.
The existence of fixed points in partially ordered metric spaces was first investigated in 2004 by Ran and Reurings [30], and then by Nieto and Rodríguez-Lopez [28]. Amini-Harandi [10] work on a new extension of the concept of partial metric space, called a metric-like space on the other hand. The concept of $b$-metric-like space which extends the notions of partial metric space, metric-like space and $b$-metric space was given by Alghamdi *et al.* [6]. They established the existence and uniqueness of fixed points in a $b$-metric-like space as well as in a partially ordered $b$-metric-like space. In addition, as an application, they derived some new fixed point and coupled fixed point results in partial metric spaces, metric-like spaces and $b$-metric spaces (see also [16,20,21,22,26]).
On the parallel lines the study of common fixed points of mappings satisfying certain contractive conditions can be employed to establish existence of solutions of certain operator equations such as differential and integral equations. Beg and Abbas [11] obtained common fixed points extending a weak contractive condition to two maps. In 2009, Doric [18] proved common fixed point theorems for generalized $(\psi,\varphi)$ - weakly contractive mappings. Abbas and Doric [5] obtained a common fixed point theorem for four maps. For more work in this direction, we refer to [11,24,32,36] and references mentioned therein. The aim of this paper is to examine more closely the structure of such spaces and obtain certain common fixed point results. In this context, we demonstrate a fundamental lemma for the convergence of sequences in $b$-metric-like spaces and by using it we prove certain common fixed point results in the setup of such spaces. Finally, examples are presented in order to verify the effectiveness and applicability of our main results.

# Preliminaries

**Definition 2.1** [17] Let $X$ be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to R^+$ is a $b$-metric if, for all $x, y, z \in X$, the following conditions are satisfied:

- $d(x,y) = 0$ if and only if $x = y$,
- $d(x,y) = d(y,x)$,
- $d(x,y) \leq s[d(x,z) + d(z,y)]$.

The pair $(X,d)$ is called a $b$-metric space with coefficient $s$.
**Definition 2.2** [27] A partial metric on a nonempty set $X$ is a mapping $p : X \times X \to R^+$ such

that for all $x,y,z \in X$ :

- $x = y$ if and only if $p(x,x) = p(x,y) = p(y,y)$ ,
- $p(x,x) \le p(x,y)$ ,
- $p(x,y) = p(y,x)$ ,
- $p(x,y) \le p(x,z) + p(z,y)$ - $p(z,z)$ .

A partial metric space is a pair $(X,p)$ such that $X$ is a nonempty set and $p$ is a partial metric.

**Definition 2.3** [34] A partial $b$ -metric on a nonempty set $X$ is a mapping $p_b$ : $X \times X \to R^+$ such that for some real number $s \ge 1$ and all $x,y,z \in X$ :

$x = y$ if and only if $p_b(x,x) = p_b(x,y) = p_b(y,y)$ ,

- $p_b(x,x) \le p_b(x,y)$ ,
- $p_b(x,y) = p_b(y,x)$ ,
- $p_b(x,y) \le s[p_b(x,z) + p_b(z,y)]$ - $p_b(z,z)$ .

A partial $b$ -metric space is a pair $(X,p_b)$ such that $X$ is a nonempty set and $p_b$ is a partial $b$-metric on $X$ . The number $s$ is called the coefficient of $(X,p_b)$ .

**Definition 2.4** [10] A metric-like on a nonempty set $X$ is a mapping $\sigma$ : $X \times X \to R^+$ such that for all $x,y,z \in X$ :

- $\sigma(x,y) = 0$ implies $x = y$ ,
- $\sigma(x,y) = \sigma(y,x)$ ,
- $\sigma(x,y) \le \sigma(x,z) + \sigma(z,y)$ .

The pair $(X,\sigma)$ is called a metric-like space.

Every partial metric space is a metric-like space. Now we give some example of metric-like spaces.

**Example 2.5** [33] Let $X = [0,1]$ . Then the mapping $\sigma_1$ : $X \times X \to R^+$ defined by $\sigma_1(x,y) = x + y$ - $xy$ is a metric-like on $X$ .

**Example 2.6** [33] Let $X = R$ , then the mappings $\sigma_i$ : $X \times X \to R^+$ ( $i \in \{2,3,4\}$ ) defined by

$$\sigma_2(x,y) = |x| + |y| + a,$$
$$\sigma_3(x,y) = |x - b| + |y - b|,$$
$$\sigma_4(x,y) = x^2 + y^2,$$

are metric-likes on $X$ , where $a \ge 0$ and $b \in R$ .

**Definition 2.7** [6] Let $X$ be a nonempty set and $s \ge 1$ be a given real number. A function $\sigma_b$ : $X \times X \to R^+$ is a $b$ -metric-like if, for all $x,y,z \in X$ the following conditions are satisfied:

- $\sigma_b(x,y) = 0$ implies $x = y$ ,
- $\sigma_b(x,y) = \sigma_b(y,x)$ ,
- $\sigma_b(x,y) \le s[\sigma_b(x,z) + \sigma_b(z,y)]$ .

A $b$ -metric-like space is a pair $(X,\sigma_b)$ such that $X$ is a nonempty set and $\sigma_b$ is a $b$ -metric-like on $X$ . The number $s$ is called the coefficient of $(X,\sigma_b)$ .

In a $b$ -metric-like space $(X,\sigma_b)$ if $x,y \in X$ and $\sigma_b(x,y) = 0$ , then $x = y$ , but the converse may not be true and $\sigma_b(x,x)$ may be positive for $x \in X$. It is clear that every partial $b$-metric space is a $b$-metric-like space with the same coefficient $s$ and every $b$-metric space is also a $b$-metric-like space with the same coefficient $s$. However, the converses need not hold.

**Example 2.8:** Let $X = R^+$, $p > 1$ a constant and $\sigma_b : X \times X \to R^+$ be defined by

$$\sigma_b(x,y) = (x+y)^p \text{ for all } x,y \in X.$$

Then $(X,\sigma_b)$ is a $b$-metric-like space with coefficient $s = 2^{p-1}$, but it is not a partial $b$-metric space. Indeed, for any $0 < y < x$ we have $\sigma_b(x,x) = (x+x)^p > (x+y)^p = \sigma_b(x,y)$, so (p $b2$) of Definition 2.3 is not satisfied.

The following propositions help us to construct some more examples of $b$-metric-like spaces.

**Proposition 2.9** Let $(X,\sigma)$ be a metric-like space and $\sigma_b(x,y) = [\sigma(x,y)]^p$, where $p > 1$ is a real number. Then $\sigma_b$ is a $b$-metric-like with coefficient $s = 2^{p-1}$.

**Proof:** The proof follows from the fact that $(a+b)^p \leq 2^{p-1}(a^p + b^p)$, where $a,b \in R^+$.

From the above proposition and Examples 2.5 and 2.6 we have the following examples of $b$-metric-like spaces.

**Example 2.10** Let $X = [0,1]$. Then the mapping $\sigma_{b1} : X \times X \to R^+$ defined by $\sigma_{b1}(x,y) = (x + y - xy)^p$, where $p > 1$ is a real number, is a $b$-metric-like on $X$ with $s = 2^{p-1}$.

**Example 2.11** Let $X = R$. Then the mappings $\sigma_{bi} : X \times X \to R^+$ ($i \in \{2,3,4\}$) defined by

$$\sigma_{b2}(x,y) = (|x| + |y| + a)^p,$$
$$\sigma_{b3}(x,y) = (|x-b| + |y-b|)^p,$$
$$\sigma_{b4}(x,y) = (x^2 + y^2)^p,$$

are $b$-metric-like on $X$, where $p > 1$, $a \geq 0$ and $b \in R$.

**Proposition 2.12** Let $X$ be a nonempty set such that $d$ and $p_b$ are $b$-metric and partial $b$-metric, respectively, $s > 1$ and $\sigma$ is a metric-like on $X$. Then the mappings $\sigma_{bi} : X \times X \to R^+$ ($i \in \{5,6,7\}$) defined by

$$\sigma_{b5}(x,y) = p_b(x,y) + d(x,y),$$
$$\sigma_{b6}(x,y) = \sigma(x,y) + p_b(x,y),$$
$$\sigma_{b7}(x,y) = \sigma(x,y) + d(x,y),$$

for all $x,y \in X$ are $b$-metric-like on $X$.

**Proof** Let $(X,p_b)$ be a partial $b$-metric space and $(X,d)$ be a $b$-metric space with $s > 1$.

Then conditions ($\sigma_b 1$), ($\sigma_b 2$) and ($\sigma_b 3$) are obvious for the function $\sigma_{b5}$. For instance, if $x,y,z \in X$ are arbitrary then, as $p_b$ is a partial $b$-metric and $d$ is a $b$-metric on $X$, we have

$$\begin{aligned}
\sigma_{b5}(x,y) &= p_b(x,y) + d(x,y) \\
&\leq s[p_b(x,z) + p_b(z,y)] - p_b(z,z) + s[d(x,z) + d(z,y)] \\
&\leq s[p_b(x,z) + d(x,z) + p_b(z,y) + d(z,y)] \\
&= s[\sigma_{b5}(x,z) + \sigma_{b5}(z,y)].
\end{aligned}$$

Therefore, ($\sigma_b 3$) is satisfied and so $(X,\sigma_{b5})$ is a $b$-metric-like space. Similarly, one can show that $(X,\sigma_{b6})$ and $(X,\sigma_{b7})$ are $b$-metric-like spaces.

From the above proposition and Examples 2.5 and 2.6 we have the following examples.

**Example 2.13** Let $X = [0,1]$. Then the mapping $\sigma_{b7} : X \times X \to R^+$ defined by $\sigma_{b7}(x,y) = x + y - xy + |x-y|^p$, where $p > 1$ is a real number, is a $b$-metric-like on $X$ with coefficient $s = 2^{p-1}$.

**Example 2.14** Let $X = R$. Then the mappings $\sigma_{bi} : X \times X \to R^+$ ($i \in \{8,9,10\}$) defined by

$$\sigma_{b8}(x,y) = |x| + |y| + a + |x-y|^p,$$
$$\sigma_{b9}(x,y) = |x-b| + |y-b| + |x-y|^p,$$
$$\sigma_{b10}(x,y) = x^2 + y^2 + |x-y|^p,$$

are $b$-metric-like on $X$ with coefficient $s = 2^{p-1}$, where $p > 1$, $a \geq 0$ and $b \in R$.

Each $b$-metric-like $\sigma_b$ on $X$ generates a topology $\tau_{\sigma_b}$ on $X$ whose base is the family of all open $\sigma_b$-balls $\{B_{\sigma_b}(x,\varepsilon) : x \in X, \quad \varepsilon > 0\}$, where $B_{\sigma_b}(x,\varepsilon) = \{y \in X : |\sigma_b(x,y) - \sigma_b(x,x)| < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Let $S$ be the class of all mappings $\beta : [0,\infty) \to [0,\frac{1}{2s^2})$ which satisfy the condition: $\beta t_n \to \frac{1}{2s^2}$ whenever $t_n \to 0$. Note that $S \neq \varphi$ as mapping $f : [0,\infty) \to [0,\frac{1}{2s^2})$ given by $fx = \frac{1}{2s^2+x}$ qualifies for a membership of $S$. Let $f$ and $g$ be two self mappings on a nonempty set $X$. If $x = fx$ (respectively, $fx = gx$ and $x = fx = gx$) for some $x$ in $X$, then $x$ is called a fixed point of $f$ (respectively, coincidence and common fixed point of $f$ and $g$). We define the following sets $F(f) = \{x \in X : x = fx\}$ and $C(f,g) = \{x \in X := xgx = fx\}$. For a complete metric space $(X,d)$, we say that $f$ is a Picard operator if the sequence $x_{n+1} = fx_n = f^n x_0$, $n = 0,1,2,\ldots$, converges to $x^*$ for each $x_0 \in X$, that is, the set $F(f) = \{x^*\}$. The set $\{x_0, fx_0, f^2x_0, f^3x_0, \cdots\}$ is called an orbit of $f$ at the point $x_0$ and denoted by $O_f(x_0)$.

In 1973, Geraghty [19] gave following generalization of a Banach fixed point theorem.

**Theorem 2.15** Let $(X,d)$ be a complete metric space and $f$ a self map on $X$. If there exists $\beta \in S$ such that $d(fx,fy) \leq \beta(d(x,y))d(x,y)$ for all $x,y \in X$. Then $f$ is a Picard operator.

In the following Harandi and Emami [10] reconsidered Theorem 2.15 in the framework of a partially ordered metric spaces.

**Theorem 2.16** Let $(X, \preccurlyeq, d)$ be a partially ordered complete metric space. Let $f : X \to X$ be an increasing mapping such that there exists an element $x_0 \in X$ with $x_0 \preccurlyeq fx_0$. If

$$d(fx,fy) \leq \alpha(d(x,y))d(x,y)$$

for each $x,y \in X$ with $x \preccurlyeq y$, where $\alpha \in S$. Then $f$ has a fixed point provided that either $f$ is continuous or $X$ is such that if an increasing sequence $\{x_n\} \to x$ in $X$; then $x_n \preccurlyeq x$, for all $n$. Besides, if for each $x,y \in X$ there exists $z \in X$ which is comparable to $x$ and $y$. Then $f$ has a unique fixed point.

**Definition 2.17** ([23]) Let $f$ and $g$ be two self-maps defined on a set $X$. Then $f$ and $g$ are said to be weakly compatible if they commute at every coincidence point.

**Definition 2.18** Let $(X,d)$ be a metric space. Then the pair $\{f,g\}$ is said to be compatible if and only if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

**Definition 2.19** Let $X$ be a nonempty set. Then $(X, d, \preccurlyeq)$ is called partially ordered $b$-metric space if and only if $d$ is a $b$-metric on a partially ordered set $(X, \preccurlyeq)$. A subset $K$ of a partially ordered set $X$ is said to be well ordered if every two elements of $K$ are comparable.

**Definition 2.20 (**[4]**)** Let $(X, \preccurlyeq)$ be a partially ordered set. A mapping $f$ is called dominating if $x \preccurlyeq fx$ for each $x$ in $X$.

**Example 2.21 (**[4]**)** Let $X = [0,1]$ be endowed with usual ordering and $f : X \to X$ be defined by $fx = \sqrt[3]{x}$. Since $x \leq x^{\frac{1}{3}} = fx$ for all $x \in X$. Therefore $f$ is a dominating map.

**Definition 2.22** Let $(X, \preccurlyeq)$ be a partially ordered set. A mapping $f$ is called dominated if $fx \preccurlyeq x$ for each $x$ in $X$.

**Example 2.23** Let $X = [0,1]$ be endowed with the usual ordering and $f : X \to X$ be defined by $fx = x^n$ for some $n \in N$. Since $fx = x^n \leq x$ for all $x \in X$. Hence, $f$ is a dominated map.

**Definition 2.24** ([9]) Let $(X, \preccurlyeq)$ be a partially ordered set. Two mappings $f, g : X \to X$ are said to be weakly increasing if $fx \preccurlyeq gfx$ and $gx \preccurlyeq fgx$ hold for all $x \in X$.

**Example 2.25** Let $X = R^+$ be endowed with usual order and usual topology. Let $f, g : X \to X$ be defined by $fx = \begin{cases} x^{\frac{1}{2}}, & \text{if } x \in [0,1] \\ x^2, & \text{if } x \in [1,\infty) \end{cases}$, $gx = \begin{cases} x, & \text{if } x \in [0,1) \\ 2x, & \text{if } x \in [1,\infty) \end{cases}$.

Then, the pair $(f, g)$ is weakly increasing where $g$ is a discontinuous mapping on $R^+$.

We also need the following definitions and propositions in the setup of $b$-metric spaces.

**Definition 2.26 (**[14]**)** Let $(X, d)$ be a $b$-metric space. Then a sequence $\{x_n\}$ in $X$ is called: (i) convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to +\infty$. In this case, we write $\lim_{n\to\infty} x_n = x$. (ii) Cauchy if and only if $d(x_n, x_m) \to 0$ as $n, m \to +\infty$.

**Proposition 2.27 (**see remark 2.1 in [14]**).** In a $b$-metric space $(X, d)$ the following assertions hold: (i) a convergent sequence has a unique limit, (ii) each convergent sequence is Cauchy, (iii) in general, a $b$-metric is not continuous.

**Definition 2.28 (**[14]**)** Let $(X, d)$ be a $b$-metric space. If $Y$ is a nonempty subset of $X$, then the closure $\overline{Y}$ of $Y$ is the set of limits of all convergent sequences of points in Y, i.e.,

$$\overline{Y} = \left\{x \in X \;:\; \text{there exists a sequence } \{x_n\} \text{ in Y such that } \lim_{n\to\infty} x_n = x\right\}.$$

Taking into account of the above definition, we have the following concepts.

**Definition 2.29** ([14]) Let $(X, d)$ be a $b$-metric space. Then a subset $Y \subset X$ is called closed if and only if for each sequence $\{x_n\}$ in $Y$ which converges to an element $x$, we have $x \in Y$ (i.e. $\overline{Y} = Y$).

**Definition 2.30 (**[14]**)** The $b$-metric space $(X, d)$ is complete if every Cauchy sequence in $X$

converges.

**Definition 2.31** [6] Let $(X,\sigma_b)$ be a $b$ -metric-like space with coefficient $s$ and let $\{x_n\}$ be any sequence in $X$ and $x \in X$ . Then: (i) The sequence $\{x_n\}$ is said to be convergent to $x$ with respect to $\tau_{\sigma_b}$ , if $lim_{n\to\infty}\,\sigma_b\,(x_n,x) = \sigma_b(x,x)$ . (ii) The sequence $\{x_n\}$ is said to be a Cauchy sequence in $(X,\sigma_b)$ if $lim_{n,m\to\infty}\,\sigma_b\,(x_n,\preccurlyeq x_m)$ exists and is finite. (iii) $(X,\sigma_b)$ is said to be a complete $b$ -metric-like space if for every Cauchy sequence $\{x_n\}$ in $X$ there exists $x \in X$ such that $\lim\limits_{m,n\to\infty} \sigma_b(x_n,x_m) = \lim\limits_{n\to\infty} \sigma_b(x_n,x) = \sigma_b(x,x)\cdot$

It is clear that the limit of a sequence in a $b$ -metric-like space is usually not unique.

**Lemma 2.32** Let $(X,\sigma_b)$ be a $b$ -metric-like space with coefficient $s > 1$ and suppose that $\{x_n\}$ and $\{y_n\}$ are convergent to $x$ and $y$ , respectively. Then we have

$$\frac{1}{s^2}\sigma_b(x,y) - \frac{1}{s}\sigma_b(x,x) - \sigma_b(y,y) \le \lim_{n\to\infty} inf\,\sigma_b(x_n,y_n) \le \lim_{n\to\infty} sup\,\sigma_b(x_n,y_n)$$
$$\le s\sigma_b(x,x) + s^2\sigma_b(y,y) + s^2\sigma_b(x,y).$$

In particular, if $\sigma_b(x,y) = 0$ , then $\lim\limits_{n\to\infty} \sigma_b(x_n,y_n) = 0$ . Moreover, for each $z \in X$ we have

$$\frac{1}{s}\sigma_b(x,z) - \sigma_b(x,x) \le \lim_{n\to\infty} inf\,\sigma_b(x_n,z) \le \lim_{n\to\infty} sup\,\sigma_b(x_n,z)$$
$$\le s\sigma_b(x,z) + s\sigma_b(x,x).$$

if $\sigma_b(x,x) = 0$ , then $\frac{1}{s}\sigma_b(x,z) \le \lim\limits_{n\to\infty} inf\,\sigma_b(x_n,z) \le \lim\limits_{n\to\infty} sup\,\sigma_b(x_n,z) \le s\sigma_b(x,z).$

**Proof** Using the triangle inequality in a $b$ -metric-like space it is easy to see that $\sigma_b(x,y) \le s\sigma_b(x,x_n) + s^2\sigma_b(x_n,y_n) + s^2\sigma_b(y_n,y)$ and

$$\sigma_b(x_n,y_n) \le s\sigma_b(x_n,x) + s^2\sigma_b(x,y) + s^2\sigma_b(y,y_n).$$

Taking the lower limit as $n \to \infty$ in the first inequality and the upper limit as $n \to \infty$ in the second inequality we obtain the first desired result. If $\sigma_b(x,y) = 0$ , then by the triangle inequality we get $\sigma_b(x,x) = 0$ and $\sigma_b(y,y) = 0$ . Therefore, we have $lim_{n\to\infty}\,\sigma_b\,(x_n,y_n) = 0$ . Similarly, using again the triangle inequality the other assertions follow.

# Contraction Conditions and Fixed Point Results

It is well known that a self-map $f$ on a metric space $(X,d)$ is said to be a Banach contraction mapping, if there exists a number $k \in [0,1)$ such that

$$d(fx,fy) \le kd(x,y)$$

for all $x,y \in X$ . A mapping $f\ :\ X \to X$ is called a quasicontraction if for some constant $\alpha \in [0,1)$ and for every $x,y \in X$, $d(fx,fy) \le \alpha max\{d(x,y),d(x,fx),d(y,fy),d(x,fy),d(y,fx)\}$.

This concept was introduced and studied by Lj. Ćirić in 1974 [15].

**Definition 2.33** Let $(\boldsymbol{X},\preccurlyeq,\boldsymbol{d_b})$ be a partially ordered $\boldsymbol{b}$ -metric-like space. If there exist $\boldsymbol{\beta} \in \boldsymbol{S}$ such that $d_b(fx,gy) \le \beta(M(x,y))N(x,y)$, for all $x,y \in X$ with $x \preccurlyeq y$ , where

$$M(x,y) = max\{d_b(x,y), d_b(x,fx), d_b(y,gy), \frac{1}{2}[d_b(x,fx) + d_b(y,gy)]\})$$

and

$$N(x,y) = \frac{1}{2s} max\{d_b(x,y), d_b(x,fx), d_b(y,gy), d_b(x,gy), d_b(y,fx)\}).$$

Then $f$ and $g$ are said to be generalized $\beta$ -order contractive mappings. We start to this section with the following theorem in which we guarantee the existence of a common fixed point of generalized $\beta$ -order contractive mappings in partially ordered $b$ -metric-like spaces.

**Definition 2.34** Let $(X, \preccurlyeq, d_b)$ be a partially ordered $b$ -metric-like space. If there exist $\beta \in S$ such that $d_b(fx, fy) \leq \beta(M(x,y))N(x,y)$, for all $x, y \in X$ with $x \preccurlyeq y$, where

$$M(x,y) = max\{d_b(x,y), d_b(x,fx), d_b(y,fy), \frac{1}{2}[d_b(x,fx) + d_b(y,fy)]\})$$

and

$$N(x,y) = \frac{1}{2s} max\{d_b(x,y), d_b(x,fx), d_b(y,fy), d_b(x,fy), d_b(y,fx)\}).$$

Then $f$ is said to be $\beta$ -order contractive mappings. We start to this section with the following theorem in which we guarantee the existence of a fixed point of $\beta$ -order contractive mappings in partially ordered $b$ -metric-like spaces.

# Common Fixed Point Result

**Theorem 3.**1 Let $(X, \preccurlyeq, d_b)$ be a partially ordered complete $b$ -metric-like space. Suppose that $f, g : X \to X$ are two weakly increasing and generalized $\beta$ - order contractive mappings. Suppose also that there exists $x_0 \in X$ such that $fx_0 \preccurlyeq gfx_0$. Assume that either $f, g$ are continuous or $X$ has a sequential limit comparison property. Then $f$ and $g$ have a unique common fixed point if and only if the set of all common fixed point is well ordered.

**Proof** Let $x_0$ be given such that $fx_0 \preccurlyeq gfx_0$. We define a sequence $\{x_n\}$ in $X$ in the following way: $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for all $n \geq 0$.

So, we have $x_1 = fx_0 \preccurlyeq gfx_0 = gx_1 = x_2$.

Analogously, $x_2 = gx_1 \preccurlyeq fgx_1 = fx_2 = x_3$. Iteratively, we obtain that

$$x_1 \preccurlyeq x_2 \preccurlyeq x_3 \preccurlyeq x_4 \preccurlyeq \cdots \preccurlyeq x_n \preccurlyeq x_{n+1} \preccurlyeq \cdots.$$

Owing to the fact that $x_{2n}$ and $x_{2n+1}$ are comparable together the assumption (1 ), we have

$$d_b(x_{2n+1}, x_{2n+2})) = d_b(fx_{2n}, gx_{2n+1})$$
$$< \frac{1}{2s^2}(\frac{1}{2s} max\{d_b(x_{2n}, x_{2n+1}), d_b(x_{2n+1}, x_{2n+2}),$$
$$s[d_b(x_{2n}, x_{2n+1}) + d_b(x_{2n+1}, x_{2n+2})], 2sd_b(x_{2n}, x_{2n+1})\}) < \frac{1}{2s^2} d_b(x_{2n}, x_{2n+1})$$
$$< d_b(x_{2n}, x_{2n+1})$$

It is clear that if we take $h = \frac{1}{2s^2}$, then the expression (2) turns into

$$d_b(x_{2n+1}, x_{2n+2}) < hd_b(x_{2n}, x_{2n+1}).$$

By using the same argument, we also get $s[\{d_b(x_{2n+2},x_{2n+1})+d_b(x_{2n+2},x_{2n+3})],2sd_b(x_{2n+2},x_{2n+1})\}) < \frac{1}{2s^2}d_b(x_{2n+2},x_{2n+1}) < d_b(x_{2n+2},x_{2n+1})$ implies that $d_b(x_{2n+3},x_{2n+2}) < hd_b(x_{2n+2},x_{2n+1})$. Hence, we conclude that $d_b(x_n,x_{n+1}) \leq hd_b(x_n,x_{n-1})$ where $h=\frac{1}{2s^2}$. Obviously, $0 \leq h < 1$. Repeating the above process, we get

$$d_b(x_n,x_{n+1}) \leq hd_b(x_n,x_{n-1}) \leq \ldots \leq h^n d_b(x_1,x_0),$$

for all $n \geq 1$, and so for $m > n$, using triangular inequality, we have

$$d_b(x_n,x_m) \leq sd_b(x_n,x_{n+1}) + s^2 d_b(x_{n+1},x_{n+2}) + \ldots + s^{m-n}d_b(x_{m-1},x_m))$$
$$= \frac{sh^n}{1-sh}d_b(x_0,x_1).$$

Since, by assumption, $h < \frac{1}{s}$, it follows that $\lim_{m,n\to\infty} d_b(x_n,x_m) = 0$. Since $X$ is complete, there exist $x^* \in X$ such that $\lim_{m,n\to\infty} d_b(x_n,x_m) = \lim_{n\to\infty} d_b(x_n,x^*) = d_b(x^*,x^*) = 0\cdot$

Now, we shall consider two alternative cases. First, suppose that $f$ is continuous, then it is clear that $x^*$ is a fixed point of $f$. Now we show that $x^* = gx^*$. Suppose, on the contrary, that $d_b(x^*,gx^*) > 0$. Regarding $x^* \preccurlyeq x^*$ together with the inequality (1), we conclude that

$$d_b(x^*,gx^*) = d_b(fx^*,gx^*) < \frac{1}{4s^3}d_b(x^*,gx^*)$$

a contradiction. Hence $x^* = gx^*$ and $x^*$ is the common fixed point of $f$ and $g$. For the second and last case, we assume that $X$ has a sequential limit comparison property. Thus, we have $x^* \preccurlyeq x_n$.Consequently, we find that $d(fx^*,x_{n+1}) = d(fx^*,gx_n) \leq \beta(M(x^*,x_n))N(x^*,x_n)$. Taking limit as $n \to \infty$, (using Lemma (2.16 {N. Hussain}), as $d_b(x^*,x^*) = 0$ ) it follows that

$$\frac{1}{s}d_b(fx^*,x^*) < \left(\frac{1}{2s^2}\right)\frac{s}{2s}d_b(fx^*,x^*),$$

and hence $d_b(fx^*,x^*) < \frac{1}{4s}d_b(fx^*,x^*)$ which shows that $fx^* = x^*$. Similarly, $gx^* = x^*$.

Here we show that the common fixed point of such mappings is unique. Suppose that the set of common fixed point is well ordered. If $\tilde{x}$ and $x^*$ are two common fixed point of $f$ and $g$, such that $\tilde{x} \neq x^*$, then from inequality (1) we have $d(x^*,\tilde{x}) = d(fx^*,g\tilde{x})$

$$\leq \beta\big(M(x^*,\tilde{x})\big)N(x^*,\tilde{x}) < \frac{1}{2s^2}d(x^*,\tilde{x}) < d(x^*,\tilde{x}),$$

a contradiction. Hence $f$ and $g$ have a unique common fixed point.

**Example 3.2** Let $X = \{1,2,3,4\}$ be a partially ordered set defined $\preccurlyeq$ on $X$ by

$$\preccurlyeq := \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}.$$

Define two self maps $f$ and $g$ such that $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 1 \end{pmatrix}$, $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \end{pmatrix}$.

It is straight forward to check that $f$ and $g$ are weakly increasing maps on $X$. Define first the $b$-metric like $d$ on $X$ by $d(1,1) = 0$, $d(2,2) = 0$, $d(3,3) = 20 = d(4,4)$, $d(x,y) =$

$d(y,x)$, $d(1,2)=\frac{1}{5}$, $d(1,3)=d(1,4)=d(2,3)=d(2,4)=10$, and $d(3,4)=5$. Then $(X,d)$ is a $b$-metric like space with $s=\frac{15}{10}$, and the function $\beta x=\frac{1}{2s^2+x}$. Note that

| $(x,y)$ | $d(fx,gy)$ | $\beta(M(x,y))N(x,y)$ |
|---|---|---|
| (1,4) | 0.2 | 0.4598 |
| (2,4) | 0.2 | 0.4598 |
| (3,3) | 0.2 | 0.2721 |
| (4,4) | 0.2 | 0.2721 |

Thus all the conditions of Theorem 3.1 are satisfied and $1$ is the unique common fixed point *of* $f$ and $g$. Note that, for $(x,y)=(3,3)$ or $(4,4)$, $d(fx,gy)\leq kd(x,y)$ is not satisfied for any value of $k$.

**Theorem 3.3** Let $(X,\preccurlyeq,d_b)$ be a partially ordered complete $b$-metric-like space. Suppose that $f : X\to X$ is dominating and $\beta$ - order contractive mappings. Suppose also that there exists $x_0\in X$ such that $fx_0\preccurlyeq gfx_0$. Assume that either $f$ is continuous or $X$ has a sequential limit comparison property. Then $f$ has a unique fixed point if and only if the set of all fixed point is well ordered.

**Proof** Let $x_0$ be given such that $x_0\preccurlyeq fx_0$. We define a sequence $\{x_n\}$ in $X$ in the following way: $x_{2n+1}=fx_{2n}$ for all $n\geq 0$. So we have $x_0\preccurlyeq fx_0=x_1$. Analogously, $x_2=fx_1$ and $x_1\preccurlyeq fx_1=x_2$ implies $x_1\preccurlyeq x_2$. Iteratively, we obtain that $x_0\preccurlyeq x_1\preccurlyeq x_2\preccurlyeq x_3\preccurlyeq x_4\preccurlyeq\cdots\preccurlyeq x_n\preccurlyeq x_{n+1}\preccurlyeq\cdots$.Owing to the fact that $x_{2n}$ and $x_{2n+1}$ are comparable together the assumption (1 ), we derive that

$$\begin{aligned}
&d_b(x_{2n+1},x_{2n+2})=d_b(fx_{2n},fx_{2n+1})\\
&<\frac{1}{2s^2}(\frac{1}{2s}max\{d_b(x_{2n},x_{2n+1}),d_b(x_{2n+1},x_{2n+2}),\\
&s[d_b(x_{2n},x_{2n+1})+d_b(x_{2n+1},x_{2n+2})],2sd_b(x_{2n},x_{2n+1})\})\\
&<\frac{1}{2s^2}d_b(x_{2n},x_{2n+1})<d_b(x_{2n},x_{2n+1})
\end{aligned}$$

It is clear that if we take $h=\frac{1}{2s^2}$, then the expression (2) turns into

$$d_b(x_{2n+1},x_{2n+2})<hd_b(x_{2n},x_{2n+1}).$$

By using the same argument, we also get

$$\begin{aligned}
&s[\{d_b(x_{2n+2},x_{2n+1})+d_b(x_{2n+2},x_{2n+3})],2sd_b(x_{2n+2},x_{2n+1})\})\\
&\qquad<\frac{1}{2s^2}(\frac{1}{2s}max\{d_b(x_{2n+2},x_{2n+1}),d_b(x_{2n+2},x_{2n+3}),\\
&s[\{d_b(x_{2n+2},x_{2n+1})+d_b(x_{2n+2},x_{2n+3})],2sd_b(x_{2n+2},x_{2n+1})\})\\
&<\frac{1}{2s^2}d_b(x_{2n+2},x_{2n+1})<d_b(x_{2n+2},x_{2n+1})
\end{aligned}$$

implies that $d_b(x_{2n+3},x_{2n+2})<hd_b(x_{2n+2},x_{2n+1})$. Hence, we conclude that $d_b(x_n,x_{n+1})\leq hd_b(x_n,x_{n-1})$, where $h=\frac{1}{2s^2}$. Obviously, $0\leq h<1$. Repeating the above process, we get

$$d_b(x_n,x_{n+1})\leq hd_b(x_n,x_{n-1})\leq\ldots\leq h^n d_b(x_1,x_0),$$

for all $n \geq 1,$ and so for $m > n,$ using triangular inequality, we have

$$d_b(x_n, x_m) \leq s d_b(x_n, x_{n+1}) + s^2 d_b(x_{n+1}, x_{n+2}) + \ldots + s^{m-n} d_b(x_{m-1}, x_m))$$

$$\leq s h^n d_b(x_0, x_1) + s^2 h^{n+1} d_b(x_0, x_1) + \ldots + s^{m-n} h^{m-1} d_b(x_0, x_1) = \frac{s h^n}{1 - s h} d_b(x_0, x_1).$$

Since, by assumption, $h < \frac{1}{s},$ it follows that $\lim_{m,n \to \infty} d_b(x_n, x_m) = 0.$ Since $X$ is complete, there exist $x^* \in X$ such that $\lim_{m,n \to \infty} d_b(x_n, x_m) = \lim_{n \to \infty} d_b(x_n, x^*) = d_b(x^*, x^*) = 0 \cdot$

Since $f$ is continuous, then it is clear that $x^*$ is a fixed point of $f$ . For the second and last case, we assume that $X$ has a sequential limit comparison property. Thus, we have $x^* \preccurlyeq x_n$ . Consequently, we find that $d(fx^*, x_{n+1}) = d(fx^*, fx_n) \leq \beta(M(x^*, x_n)) N(x^*, x_n).$

Taking limit as $n \to \infty$ , (using Lemma (2.16 {N. Hussain}), as $d_b(x^*, x^*) = 0$ ) it follows that

$$\frac{1}{s} d_b(fx^*, x^*) < (\frac{1}{2s^2}) \frac{s}{2s} d_b(fx^*, x^*),$$

and hence $d_b(fx^*, x^*) < \frac{1}{4s} d_b(fx^*, x^*)$ which shows that $fx^* = x^*$ .

Here we show that the fixed point of such mappings is unique. Suppose that the set of fixed point is well ordered. If $\tilde{x}$ and $x^*$ are two fixed point of $f$ , such that $\tilde{x} \neq x^*$ , then from inequality (1) we have $d(x^*, \tilde{x}) = d(fx^*, f\tilde{x}) \leq \beta(M(x^*, \tilde{x})) N(x^*, \tilde{x}) < \frac{1}{2s^2} d(x^*, \tilde{x}) < d(x^*, \tilde{x}),$ a contradiction. Hence $f$ has a unique common fixed point.

# References


1) M. Abbas and B. E. Rhoades, Fixed and periodic point results in cone metric spaces, Appl. Math. Letters, 22 (2009) 511--515.
2) M. Abbas, M. A. Khamsi and A. R. Khan, Common fixed point and invariant approximation in hyperbolic ordered metric spaces, Fixed Point Theory and Applications 2011, 25.
3) M. Abbas, A. R. Khan and S. Z. Nemeth, Complementarity problems via common fixed points in vector lattices, Fixed Point Theory and Applications 2012, 60.
4) M. Abbas, T. Nazir and S. Radenovic, Common fixed points of four maps in partially ordered metric spaces, Applied Mathematics Letters 24 (2011) 1520-1526.
5) M. Abbas and D. Doric, Common fixed point theorem for four mappings satisfying generalized weak contractive condition, Faculty of Sciences and Maths., Uni. of Nis, Serbia, Filomat 24 (2) (2010) 1--10.
6) M.A. Alghamdi, N. Hussain and P. Salimi, Fixed point and coupled fixed point theorems on $b$ - metric-like spaces, Fixed Point Theory and Appl., 2013, 2013:402.
7) A. Aghajani, M. Abbas and J.R. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered $b$ -metric spaces, to appear in Math. Slovaca.
8) Ya. I. Alber, S. G. Delabriere, Principle of weakly contractive maps in Hilbert spaces, New Results in Operator Theory and Its Applications Operator Theory, Advances and

Applications Volume 98, (1997) 7-22.
9) I. Altun, B. Damjanovic and D. Djoric, Fixed point and common fixed point theorems on ordered cone metric spaces, Appl. Math. Lett. 23 (2010) 310- 316.
10) A. A. Harandi and H. Emami, A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations, Nonlinear Analysis. 72 (5) (2010), 2238-2242.
11) I. Beg and M. Abbas, Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition, Fixed Point Theory and Applications, vol. (2006) 1-7.
12) M. Boriceanu, Strict fixed point theorems for multivalued operators in $b$ -metric spaces, International Journal of Modern Mathematics. 4(3) (2009), 285-301.
13) M. Boriceanu, Fixed point theory for multivalued generalized contraction on a set with two $b$ -metrics, Studia Univ. Babes--Bolyai, Mathematica, Volume LIV, Number 3, (2009).
14) M. Boriceanu, M. Bota and A. Petrusel, Multivalued fractals in $b$ -metric spaces, Cent. Eur. J. Math. 8(2) (2010), 367-377.
15) Lj. B. Ćirić, A generalization of Banach's contraction principle, Proc. Amer. Math. Soc., 45 (1974) 267--273.
16) Lj. Ćirić, M. Abbas, R. Saadati and N. Hussain, Common fixed points of almost generalized contractive mappings in ordered metric spaces, Appl. Math. Comput., 217 (2011) 5784--5789.
17) S. Czerwik, Nonlinear set-valued contraction mappings in $b$ -metric spaces, Atti Sem. Mat. Fis. Univ. Modena. 46 (2) (1998), 263-276.
18) D. Doric, Common fixed point for generalized $(\psi, \phi)$ -weak contractions, Appl. Math. Lett. 22 (2009), 1896-1900.
19) M. Geraghty, On contractive mappings, Proc. Amer. Math. Soc. 40 (1973) 604-608.
20) N. Hussain, D. DJorić, Z. Kadelburg and S. Radenović, Suzuki-type fixed point results in metric type spaces, Fixed Point Theory and Applications, 2012, 2012:126.
21) M. H. Shah, Suzana Simic, N. Hussain, A. Sretenović, and S. Radenović, Common fixed points theorems for occasionally weakly Compatible pairs on cone metric type spaces, J. Comput. Anal. Appl.,14 (2012), 290--297.
22) N. Hussain and M. H. Shah, KKM mappings in cone $b$ -metric spaces, Computers Math. Appl., 62(2011), 1677-1684.
23) G. Jungck, Compatible mappings and common fixed points, Int. J. Math. Math. Sci. 9 (4) (1986), 771-779.
24) E. Karapnar and I. S. Yuce, Fixed Point Theory for Cyclic Generalized Weak $\phi$ -Contraction on Partial Metric Spaces, Abstr. Appl. Anal. Volume 2012 (2012).
25) M. A. Khamsi, Remarks on cone metric spaces and fixed point theorems of contractive mappings, Fixed Point Theory Appl., 2010, Article ID 315398, 7 pages, doi:10.1155/2010/315398.

26) N. Shobkolaei, S. Sedghi, J. R. Roshan, and N. Hussain, Suzuki type fixed point results in metric-like spaces, J. Function Spaces Appl., Vol. 2013, Article ID 143686, 9 pages, 2013.
27) S. Shukla, Partial $b$ -metric spaces and fixed point theorems, Mediterr. J. Math. DOI 10.1007/s00009-013-0327-4.
28) S. L. Singh and B. Prasad, Some coincidence theorems and stability of iterative proceders, Comput. Math. Appl. 55 (2008), 2512-2520.
29) Q. Zhang and Y. Song, Fixed point theory for generalized $\phi$ -weak contraction, Appl. Math. Lett. 22 (2009), 75-78.
30) M. A. Khamsi and N. Hussain, KKM mappings in metric type spaces, Nonlinear Analysis, 73 (2010) 3123-3129.
31) S. G. Matthews, Partial metric topology, in: General Topology and its Applications, Proc. 8th Summer Conf., Queen's College, 1992, in: Annals of the New York Academy of Sciences, vol. 728, 1994, pp. 183--197.
32) J. J. Nieto and R. R. Lopez, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. Order 22 (2005), 223-239.
33) M. Pacurar, Sequences of almost contractions and fixed points in $b$ -metric spaces, Analele Universitatii de Vest, Timisoara Seria Matematica Informatica XLVIII. 3 (2010), 125-137.
34) A. C. M. Ran and M. C. B. REURINGS, A fixed point theorem in partially ordered sets and some application to matrix equations, Proc. Amer. Math. Soc. 132 (2004), 1435-1443.
35) A. Razani, V. Parvaneh, and M. Abbas, A common fixed point for generalized $(\psi,\phi)_{f,g}$ -weak contractions, Ukrainian Math. J., 63(11) 2012.
36) B. E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Anal. 47 (2001), 2683-2693